Stochastic Analysis
Problem Set 1

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Throughout this document, every random object is assumed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}$ denoting mathematical expectation with respect to $\mathbb{P}$. We focus on the following definition:

Definition 1 (a) Let $(A, \mathcal{A}, \mu)$ be a σ-finite measure space, such that $(A, \mathcal{A})$ is a Polish space and μ has no atoms (note that, in this case, $\mathcal{A}$ is necessarily generated by a countable collection of sets). Write $L^2(\mu) := L^2_{\mathbb{R}}(A, \mathcal{A}, \mu)$. Set $\mathcal{A}_0 := \{B \in \mathcal{A} : \mu(B) < \infty\}$. A **Gaussian measure** on $(A, \mathcal{A})$ with **intensity** μ is a centered Gaussian family of the type

$$\mathbf{G} = \{G(B) : B \in \mathcal{A}_0\},$$

such that $\mathbb{E}[G(B)G(C)] = \mu(B \cap C)$ for every $B, C \in \mathcal{A}_0$.

(b) Let $\mathscr{H}$ be a real separable Hilbert space. An **isonormal Gaussian process** over $\mathscr{H}$ is a centered Gaussian family of the type

$$\mathbf{X} = \{X(h) : h \in \mathscr{H}\},$$

such that, for every $h, h' \in \mathscr{H}$, $\mathbb{E}[X(h)X(h')] = \langle h, h' \rangle_{\mathscr{H}}$. (**NB**: *in the classroom we will prove the existence of isonormal Gaussian processes in the special case $\mathscr{H} = L^2(\mu)$ — but the proof extends almost verbatim to this more general setting*).

(c) We keep the notation of the previous Point (b), and let $\mathscr{H}_1$ be a proper real Hilbert subspace of $\mathscr{H}$. The $\mathscr{H}_1$-**pinned process** associated with X is the centered Gaussian family

$$\mathbf{x} = \{x(h) : h \in \mathscr{H}\},$$

where

$$x(h) := X\Big(\mathrm{proj}(h \,|\, \mathscr{H}_1^{\perp})\Big), \quad h \in \mathscr{H}.$$

For the sake of completeness, we recall the following basic result, that you might need in order to solve some of the exercises below (it will be useful for the whole semester, indeed).

Lemma 2 *Let $\{X_j : j \in J\}$ and $\{Y_k : k \in K\}$ be two arbitrary collections of random variables, and define*

$$\mathcal{X} = \sigma\{X_j : j \in J\}, \quad \mathcal{Y} = \sigma\{Y_k : k \in K\}.$$

Then, the following two assertions are equivalent:

(i) *$\mathcal{X}$ and $\mathcal{Y}$ are independent;*

(ii) *For any integers $m, n \geq 1$ and for any choice of*

$$j_1, ..., j_m \in J \quad \textit{and} \quad k_1, ..., k_n \in K,$$

the vectors $(X_{j_1}, ..., X_{j_m})$ and $(Y_{k_1}, ..., Y_{k_n})$ are independent.

Proof. The fact that (i) ⇒ (ii) is trivial, so we will only show the implication (ii) ⇒ (i). To accomplish this, fix $j_1, ..., j_m \in J$ as well as Borel sets $A_1, ..., A_m$, and observe that, if (ii) is in order, then the two mappings, from $\mathcal{Y}$ into $[0, 1]$,

$$B \mapsto \mathbb{P}[\{X_{j_1} \in A_1, ..., X_{j_m} \in A_m\} \cap B], \quad \text{and} \quad B \mapsto \mathbb{P}[\{X_{j_1} \in A_1, ..., X_{j_m} \in A_m\}] \times \mathbb{P}[B],$$

define two finite measures on $\mathcal{Y}$, that coincide on *cylindrical sets*, that is, on sets of the form

$$B = \{Y_{k_1} \in C_1, ..., Y_{k_n} \in C_n\},$$

where $k_1, ..., k_n \in K$ and $C_1, ..., C_n$ are arbitrary indices and arbitrary Borel sets, respectively. Since cylindrical sets of the above form constitute a π-system generating $\mathcal{Y}$, we conclude that

$$\mathbb{P}[\{X_{j_1} \in A_1, ..., X_{j_m} \in A_m\} \cap B] = \mathbb{P}[\{X_{j_1} \in A_1, ..., X_{j_m} \in A_m\}] \times \mathbb{P}[B],$$

for every $B \in \mathcal{Y}$. Similarly, this last relation yields immediately that two mappings, from $\mathcal{X}$ into $[0, 1]$,

$$A \mapsto \mathbb{P}[A \cap B], \quad \text{and} \quad A \mapsto \mathbb{P}[A] \times \mathbb{P}[B],$$

define two finite measures on $\mathcal{X}$, that coincide on cylindrical sets of the form

$$A = \{X_{j_1} \in A_1, ..., X_{j_m} \in A_m\},$$

where $j_1, ..., j_m \in J$ and $A_1, ..., A_m \in \mathscr{B}(\mathbb{R})$ are arbitrary. By the same reasoning as above, this implies that

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B],$$

for every $A \in \mathcal{X}$ and every $B \in \mathcal{Y}$, and the desired conclusion is therefore achieved. ■

EXERCISES-I:

1. Prove that, for all $x \in A$, $G(\{x\}) = 0$, a.s.-$\mathbb{P}$.

2. Show that, if $A, B \in \mathcal{A}_0$, then $G(A \cup B) + G(A \cap B) = G(A) + G(B)$, a.s.-$\mathbb{P}$.

3. Prove that, if $\{B_n : n \geq 1\} \subseteq \mathcal{A}_0$ is a collection of pairwise disjoint sets such that $B \in \mathcal{A}_0$, then

$$G(B) = \sum_{n=1}^{\infty} G(B_n), \quad a.s. - \mathbb{P},$$

where the series on the right-hand side converges in L^2.

4. Let $B \in \mathcal{A}_0$ and let

$$B_n := \{B_{n,1}, ..., B_{n,k_n}\}, \quad n \geq 1,$$

be a sequence of measurable partitions of B such that, as $n \to \infty$, $k_n \to \infty$, and

$$\tau_n := \max_{j=1,...,k_n} \mu(B_{n,j}) \longrightarrow 0.$$

Show that, as $n \to \infty$,

$$\sum_{j=1}^{k_n} G(B_{j,n})^2 \longrightarrow \mu(B),$$

in $L^2(\mathbb{P})$. What is the limit of $\sum_{j=1}^{k_n} \mu(B_{j,n})^2$?

5. Prove that, for every choice of $f, g \in \mathscr{H}$ and $a \in \mathbb{R}$,

$$X(af + g) = aX(f) + X(g), \quad \text{a.s.-}\mathbb{P}$$

6. Let $\mathscr{H}_1$ be a proper Hilbert subspace of $\mathscr{H}$, and define

$$\mathcal{U} = \sigma(X(h) : h \in \mathscr{H}_1), \quad \mathcal{V} = \sigma(X(h) : h \in \mathscr{H}_1^{\perp}).$$

Prove that $\mathcal{U}$ and $\mathcal{V}$ are independent. Let $\mathbf{x}$ be the $\mathscr{H}_1$-pinned process introduced in Definition 1-(c), and write

$$\mathcal{V} = \sigma(X(h) : h \in \mathscr{H}_1^{\perp}),$$

as before. Deduce that, for every $h \in \mathscr{H}$,

$$\mathbb{E}[X(h) \,|\, \mathcal{V}] = x(h).$$

We recall that a **stochastic process** is a collection of random variables of the type $\{X_t : t \in I\}$, where I is a subset of the real line. A **Gaussian process** is a stochastic process that is also a Gaussian family, in the sense explained in the course.

Definition 3 Let I indicate either the half-line $\mathbb{R}_+ = [0, \infty)$, or an interval of the type $[0, T]$, with $0 < T < \infty$.

1. A (standard) **pre-Brownian motion** on I is a Gaussian process of the type $W = \{W_t : t \in I\}$, verifying the following properties: (i) $W_0 = 0$, (ii) $\mathbb{E}[W_t] = 0$, for every $t > 0$, and (iii) $\mathbf{Cov}(W_s, W_t) = \min(s, t)$.

2. Let W be a pre-Brownian motion on $\mathbb{R}_+$, and fix a finite $T > 0$. For every $t \in [0, T]$, write

$$b_t^T = W_t - \frac{t}{T} W_T.$$

We call $b^T = \{b_t^T = t \in [0, T]\}$ the **(pre)-Brownian bridge** of length T, from zero to zero.

EXERCISES-II:

1. Let X be an isonormal Gaussian process over $\mathscr{H} := L^2(\mathbb{R}_+, \mathscr{B}(\mathscr{R}_+), dt)$. Show that the stochastic process $\{X_t : t \geq 0\}$, defined by $X_0 = 0$ and $X_t = X(\mathbf{1}_{[0,t]})$, $t > 0$ is a pre-Brownian motion.

2. Let X be an isonormal process over $\mathscr{H} := L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), dt)$, and fix $T > 0$. Find an appropriate Hilbert subspace $\mathscr{H}_1 \subset \mathscr{H}$, such that the process

$$\{x(\mathbf{1}_{[0,t]}) : t \in [0, T]\},$$

(where $\mathbf{x} = \{x(h) : h \in \mathscr{H}\}$ is the associated $\mathscr{H}_1$-pinned process) has the same finite-dimensional distributions as b^T.

3. Prove that a stochastic process $W = \{W_t : t \in I\}$ is a pre-Brownain motion if and only if the following propreties hold: (i) $W_0 = 0$, (ii) for every $0 \leq s < t$, $W_t - W_s \sim \mathscr{N}(0, t - s)$, and (iii) for every $0 \leq s < t$, the increment $W_t - W_s$ is stochastically independent of $\sigma(W_u : u \leq s)$.

4. Let $W = \{W_t : t \geq 0\}$ be a pre-Brownian motion, and fix $0 < t_1 < \cdots < t_n$. Compute the probabiity densities of the following two random vectors:

$$(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}), \quad \text{and} \quad (W_{t_1}, W_{t_2}, \ldots, W_{t_n})$$

5. Let $W = \{W_t : t \geq 0\}$ be a pre-Brownian motion. Prove that the following four processes are also pre-Brownian motions:

$$\begin{aligned}
A_t &= c^{-1} W_{c^2 t}, \quad t \geq 0 \quad \text{(for any choice of } c > 0\text{)}, \\
B_t &= W_{T-t} - W_t, \quad t \in [0, T] \quad \text{(for any choice of } 0 < T < \infty\text{)}, \\
C_t &= -W_t, \quad t \geq 0 \\
W_t^z &= W_{z+t} - W_z, \quad t \geq 0 \quad \text{(for any choice of } z > 0\text{)}.
\end{aligned}$$

Show tat, for every fixed $z > 0$, the process W^z is independent of the σ-field

$$\sigma(W_r : r \leq z).$$