Stochastic Analysis
Problem Set 2

G. Peccati, P. Perruchaud

Every random object considered below is defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Problem 1 – Fractional Brownian Motion

Let $H \in (0,1)$. A *fractional Brownian motion with Hurst parameter* H is a real-valued Gaussian process

$$W^H = \left\{W_t^H : t \geq 0\right\}$$

verifying.

$$\begin{aligned} \forall t &\geq 0, \quad \mathbb{E}\left(W_t^H\right) = 0 \\ \forall s,t &\geq 0, \quad \mathbb{E}\left(W_t^H \times W_s^H\right) = \Gamma^H(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \end{aligned}$$

(1) Show that Γ^H is a well-defined covariance function, and therefore that fractional Brownian motion exists. For every $H \in (0,1)$ and every $0 \leq s < t < s' < t' < +\infty$, determine the law of the vector

$$\left(W_t^H - W_s^H, W_{t'}^H - W_{s'}^H\right).$$

(2) What is $W^{\frac{1}{2}}$?

(3) Show that, for every $c > 0$,

$$W^H \overset{\text{law}}{=} \left\{c^H \times W_{c^{-1}t}^H : t \geq 0\right\}.$$

(4) Show that, for every $H \in (0,1)$, W^H admits a continuous modification. Is the modification α-Hölder continuous for some α? Use your computations to deduce that, with probability one, W^H is γ-Hölder continuous, for every $\gamma < H$.

Problem 2 – Brownian paths are not differentiable

"*With horror and dread do I turn away from this miserable plague: functions that have no derivatives...* " (**Ch. Hermite**, in a letter to **I.J. Stieltjes**).

Let $W = \{W_t : t \geq 0\}$ be a standard Brownian motion initialized at zero, defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a continuous modification of W. We shall prove that, for every $t^* \geq 0$, the mapping $t \mapsto W_t(\omega)$ is not differentiable in t^*, a.s. $-\mathbb{P}(d\omega)$.

(1) Fix $A > 0$. Show that

$$\lim_{t \downarrow 0} \mathbb{P}\left\{\left|W_t(\omega)\right| \leq At\right\} = 0.$$

(2) Use (1) to prove that, for fixed $A > 0$ and $\varepsilon > 0$,

$$\mathbb{P}\left\{\left|W_t(\omega)\right| \leq At, \quad \forall t \in [0, \varepsilon)\right\} = 0.$$

(3) Use (2) to prove that, with probability one, W_t is not differentiable in $t = 0$.

(4) Deduce from (3) that, for every $t^* > 0$, with probability one the mapping $t \mapsto W_t(\omega)$ is not differentiable in t^*.

We can prove much more, namely that

$$t \mapsto W_t(\omega) \text{ is almost surely not differentiable for every } t.$$

This result, proved by Paley, Wiener and Zygmund (1933) is explained e.g. in Chapter 1 of the book by Mörters-Peres indicated at the beginning of this course.

Problem 3 – Let $\{W_t : t \geq 0\}$ be a standard Brownian motion initialized at zero. Define the process $\beta = \{\beta_t : t \geq 0\}$ as follows: $\beta_0 = 0$ and

$$\beta_t = t W_{1/t}, \quad t > 0.$$

Prove that β is indistinguishable from a standard Brownian motion.

Problem 4 – Let $X = \{X_t : t \in [0, T]\}$ be a standard Brownian motion on $[0, T]$. We assume that $\mathcal{F} = \sigma(X)$. As usual, we write $\mathcal{F}_t = \sigma\{X_s : s \leq t\}$, $0 \leq t \leq T$, to indicate the filtration generated by X. We also introduce the enlarged filtration

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(X_T), \quad 0 \leq t \leq T.$$

The aim of this exercise is to study some properties of X, as a process adapted to $\mathcal{G}_t$.

(1) Prove that the random variable

$$E\left[\int_0^T \left|\frac{X_T - X_s}{T - s}\right| ds\right] < \infty,$$

and therefore that the random variable $\int_0^T \frac{X_T - X_s}{T-s} ds$ is finite, P-almost surely. In what follows we shall set, by convention, $\int_0^T \frac{X_T(\omega) - X_s(\omega)}{T-s} ds = 0$, on the exceptional set of those $\omega \in \Omega$ such that $\int_0^T \left|\frac{X_T(\omega) - X_s(\omega)}{T-s}\right| ds = \infty$.

(2) We define the stochastic process

$$X_t^{(T)} := X_t - \int_0^t \frac{X_T - X_s}{T - s} ds, \quad 0 \leq t \leq T.$$

Prove that $X^{(T)}$ is Gaussian, centered and $\mathcal{G}_t$-adapted. Prove that $X^{(T)}$ is stochastically independent of X_T.

(3) Let $0 \leq s < t \leq T$. Prove that the increment $X_t^{(T)} - X_s^{(T)}$ is independent of $\mathcal{G}_s$.