Quiz 3

This quiz is a take-home exam. Duration is 24 hours, from April 21st 3:30PM to April 22nd 3:30PM. The work is to be handed back on Sakai, preferably in `.pdf` format. You may email me questions at any time.

Collaboration is prohibited, between students as well as with other parties. The following resources are authorised: personal notes, the course videos, the documents available on the course website (including the course textbook), Wikipedia in English. Unless otherwise specified, other resources are prohibited. Basic calculators, and the use of the computer for similar operations, are allowed; anything more sophisticated is forbidden.[1]

As usual, the Notre Dame Code of Honor is in effect for this midterm.

Out of the four following exercises, you may ignore one. Should you choose not to, only the best three will count in the grade.

Exercise 1

Let X and Y by continuous random variables with the joint probability density function

$$p(x,y) = \begin{cases} \frac{x^2+\sin(\pi y)}{C} & \text{for } x, y \in [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

for some constant $C > 0$.

1. What is the value of C?
2. What is the correlation of X and Y? Are X and Y independent?

[1] Computing $\sqrt{2} + \ln(17 - e)$ is fine; plotting $f(x) = x^2 - 3$ isn't.

Exercise 2

We consider a button wired to a tally counter, but through a random device. Initially, the counter indicates zero. Then if at some point the counter shows k when the button is pressed, the counter is incremented only with probability 2^{-k}, otherwise nothing happens. For instance, the first time the button is pressed, the counter indicates zero so it increases with probability 1. The second time, the counter is one so it goes to either 1 or 2 with equal probability 1/2.

We write X_n for what the counter shows after the button is pressed n times (so X_n is a random variable and $X_0 = 0$). We denote their probability mass function by p_n; since the variables take non-negative integer values, we have for instance $p_n(-2) = 0$ for all n.

1. What is p_2?

2. Suppose p_n is known. Express the probability mass function of (X_n, X_{n+1}) using p_n but not p_{n+1}.

3. Let Y be a random variable with values in $\mathbb{N}$. What is the value of the following sum?

$$\sum_{k\geq 0} p_Y(k-1)$$

4. Show that

$$\mathbb{E}\left[2^{X_{n+1}}\right] = \mathbb{E}\left[2^{X_n}\right] + 1.$$

   For instance, you can see X_{n+1} as a function of (X_n, X_{n+1})

5. Use induction to show that

$$\mathbb{E}\left[2^{X_n} - 1\right] = n.$$

It is possible to show that $2^{X_n}/n$ is close to one with fairly high probability, so the approximation $n \simeq 2^{X_n}$ is good most of the time.

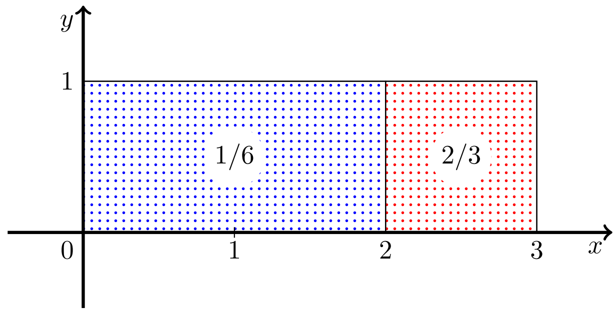
This kind of technique is used in computer science, when we want to save a humongous integer in memory but we do not need a good precision, or if we don't want to write on that value too often. For instance, if we want to detect a DDoS attack on a server, we can count the requests (they are many) and trigger an alarm when we reach a certain critical threshold; the fact that the number is not precisely measured is not a problem, provided it is roughly accurate.

A usual integer is stored on 32 bits in a computer, which enables us to count up to 4,294,967,295. Because we do not store n but rather $X_n \simeq \log_2(n)$, we can go up to numbers of order $2^{4,294,967,295}$. These are numbers with about a billion of digits, and it's safe to say they make no physical sense. For instance, the ratio between the size of the observable universe and that of an atom is believed to have about 37 digits.

Exercise 3

Let X and Y be two variables which admit a joint probability density function p given by

$$p(x,y) = \begin{cases} 1/6 & \text{if } (x,y) \in (0,2) \times (0,1), \\ 2/3 & \text{if } (x,y) \in (2,3) \times (0,1), \\ 0 & \text{else.} \end{cases}$$



We imagine that X and Y are the price in a few months' time of some raw product. An investment company wants to choose between two strategies A and B, which have been designed to give a respective (random) profit e^{-X} and Y^2.

1. What are the probability density functions of the marginals X and Y?
2. What are the expectations of e^{-X} and Y^2? Which strategy is better in average?
3. Let $\Omega \subset \mathbb{R}^2$ be the subset of all (x, y) such that the strategy A is better than the strategy B when $X = x$, $Y = y$. Keeping in mind that plotters are not allowed, draw Ω roughly on a graph. Indicate the values of p on the same graph.
4. What is the probability that the strategy A is better than the strategy B?

Exercise 4

Let X be a variable with distribution $\mathcal{E}xp(2)$. Define

$$f(x) = \frac{2X}{1 + X^2}$$

and $Y = f(X)$.

1. Find the critical points of f and its limit at $+\infty$. Draw roughly the graph of f over $\mathbb{R}_+$.
2. Find all points $x \geq 0$ such that $f(x) \leq 1/\sqrt{2}$.
3. What is the density of Y?