Quiz 3 *bis*

This exercise is a follow up on Quiz 3, and will be graded as described in my email dated 4/24, 2:00PM EST. The work is to be handed back before April 27th 11:59PM on Sakai, preferably in `.pdf` *format. You may email me questions at any time.*

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As usual, the Notre Dame Code of Honor is in effect for this work.

Exercise 1

Let X and Y by continuous random variables with the joint probability density function

$$p(x,y) = \begin{cases} \frac{1}{C} \cdot \left(\ln(x) + \frac{1}{y^4}\right) & \text{for } x, y \in [1,e] \times [1,2], \\ 0 & \text{otherwise,} \end{cases}$$

for some constant $C > 0$.

1. What is the value of C?
2. What is the variance of X?
3. What is the covariance of X and Y? Are X and Y independent?

[1] Computing $\sqrt{2} + \ln(17 - e)$ is fine; plotting $f(x) = x^2 - 3$ isn't.

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Exercise 2

We consider a button wired to a numeric display through a random device. Initially, the counter indicates one. Each time the button is pressed, the counter is incremented doubled with probability 2/3, and halved with probability 1/3. For instance, the first time the button is pressed, the counter was one so it becomes 2 with probability 2/3 or 1/2 with probability 1/3.

We write X_n for what the counter shows after the button is pressed n times (so X_n is a random variable and $X_0 = 0$). We denote their probability mass function by p_n. Note that X_n is always a power of two: $X_n = 2^k$, for k possibly negative.

1. What is p_2?
2. Suppose p_n is known. Express the probability mass function of (X_n, X_{n+1}) using p_n but not p_{n+1}.
3. Let Y be a random variable with values of the form 2^k for k relative integers. Show that for any function f,
$$\sum_{k\in\mathbb{Z}} f(2^k)p_Y(2^{k-1}) = \mathbb{E}[f(2Y)].$$
4. Show that
$$\mathbb{E}\left[\ln(X_{n+1})\right] = \mathbb{E}\left[\ln(X_n)\right] + \frac{1}{3}\ln(2).$$
For instance, you can see X_{n+1} as a function of (X_n, X_{n+1})
5. Use induction to show that
$$\mathbb{E}\left[\ln(X_n)\right] = \frac{n}{3}\ln(2).$$

[1] Computing $\sqrt{2} + \ln(17 - e)$ is fine; plotting $f(x) = x^2 - 3$ isn't.

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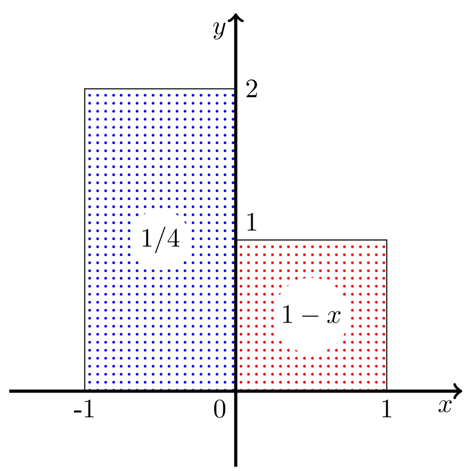
Exercise 3

Let X and Y be two variables which admit a joint probability density function p given by

$$p(x,y) = \begin{cases} 1/4 & \text{if } (x,y) \in (-1,0) \times (0,2), \\ 1-x & \text{if } (x,y) \in (0,1) \times (0,1), \\ 0 & \text{else.} \end{cases}$$



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We imagine that X and Y are the price in a few months' time of some raw product. An investment company wants to choose between two strategies A and B, which have been designed to give a respective (random) profit XY and $2Y - 1$.

1. What are the probability density functions of the marginals X and Y?

2. What are the expectations of XY and $2Y - 1$? Which strategy is better in average?

3. Let $\Omega \subset \mathbb{R}^2$ be the subset of all (x, y) such that the strategy A is better than the strategy B when $X = x$, $Y = y$. Keeping in mind that plotters are not allowed, draw Ω roughly on a graph. Indicate the values of p on the same graph.

4. What is the probability that the strategy A is better than the strategy B?

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Exercise 4

Let X be a variable with distribution $\mathcal{G}amma(2,1)$. Recall that by definition, its density is

$$p_X(x) = \begin{cases} \frac{1}{C} \cdot x^{2-1} e^{-x} & \text{for } x > 0, \\ 0 & \text{else} \end{cases}$$

for some constant C. You may use C directly in your answers, without giving its value.[2]

Define

$$f(x) = -\left(x + \frac{1}{x} \right)$$

and $Y = f(X)$.

1. Find the critical points of f and its limit at $+\infty$ and 0. Draw roughly the graph of f over $\mathbb{R}_+$.
2. Find all points $x \geq 0$ such that $f(x) \leq -4$.
3. What is the density of Y?

[1] Computing $\sqrt{2} + \ln(17 - e)$ is fine; plotting $f(x) = x^2 - 3$ isn't.

[2] If you really want to know, you can deduce the value of C from Appendix A for the textbook. It is a simple constant.