Review sheet

Preparation for Midterm 2

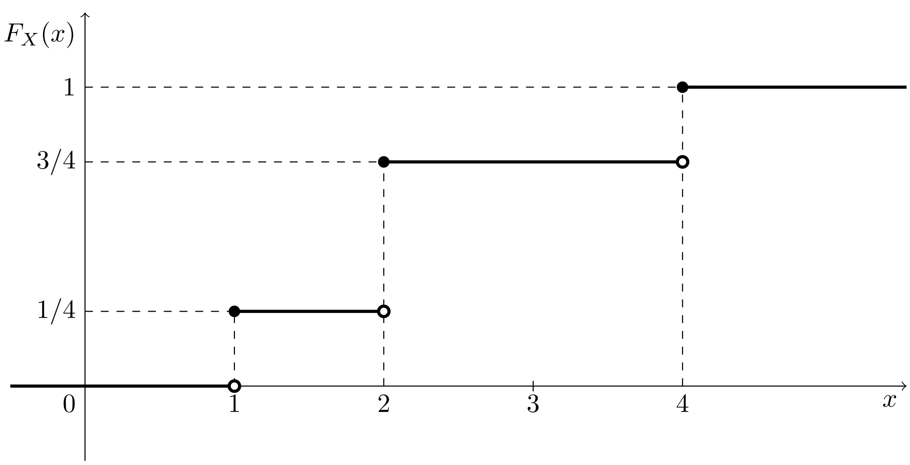
Expectation/variance/moments of discrete variables

Exercise A. Let $X \sim \mathcal{B}in(2, 1/3)$. Compute the expectation

$$\mathbb{E}\left[\frac{1}{1+X}\right].$$

Exercise B. Compute the expectation and the variance of the following random variables.

1. X is a variable with the following cumulative distribution function:



2. X takes the following values with the following probabilities:

$$\mathbb{P}(X=-2) = \frac{1}{5}, \qquad \mathbb{P}(X=0) = \frac{3}{10}, \qquad \mathbb{P}(X=1) = \frac{1}{6}, \qquad \mathbb{P}(X=3) = \frac{1}{3}.$$

Exercise C. A cook hesitates between two recipes, a high-end one that could sell for \$20 each, or a more conventional priced at \$12. However, the fashionable one would be more complicated to do, with 2% of the dishes failing and having to be thrown out, while the easier one would only fail half as often (1% of dishes thrown out).

1. What would be the best strategy if the restaurant wanted to maximise the long-term profit?
2. What would be the best strategy if the restaurant wanted to minimise the uncertainty?

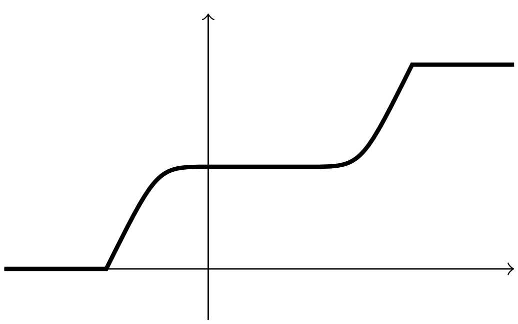
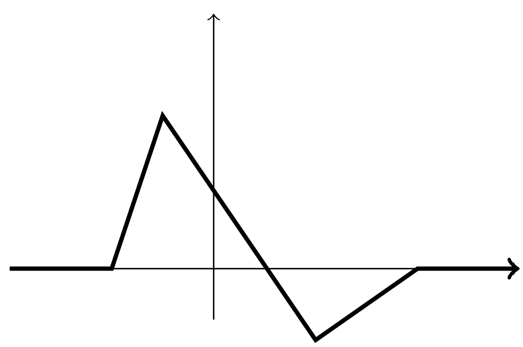
Exercise D.

1. Let X_n be a sequence of random variables with distribution $\mathcal{B}in\left(n, \frac{1}{n}\right)$. Give an upper bound on $\mathbb{P}(X_n \geq 10)$ that does not depend on n.[1]
2. Let X_n be a sequence of random variables with distribution $\mathcal{B}in\left(n, \frac{1}{2}\right)$. Give an upper bound on
$$\mathbb{P}\left(\left|\frac{1}{n}X_n - \frac{1}{2}\right| \geq \frac{1}{10}\right)$$
that goes to zero as n goes to infinity.

Continuous random variables

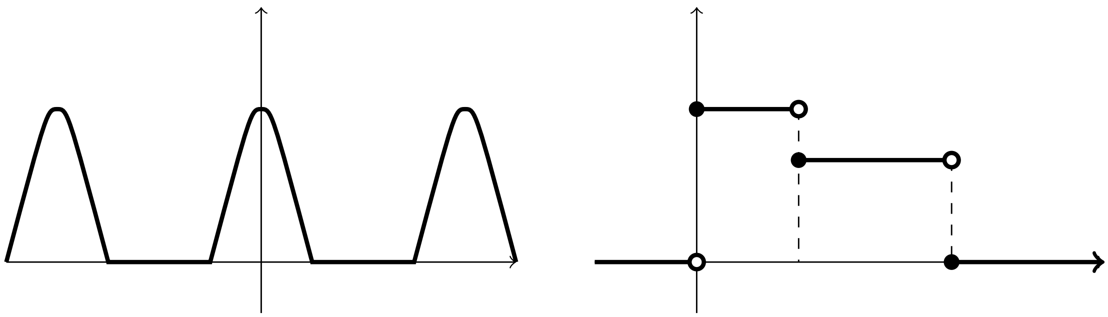
Exercise E. Below are 6 graphs. For each of them,

- determine if it is possible for the graph to be that of a cumulative distribution function or a density function;
- if it is the graph of a density, sketch the graph of the associated cumulative distribution function;
- if it is the graph of a cumulative distribution function, sketch the associated probability density function (if the variable is continuous) or probability mass function (if the variable is discrete).

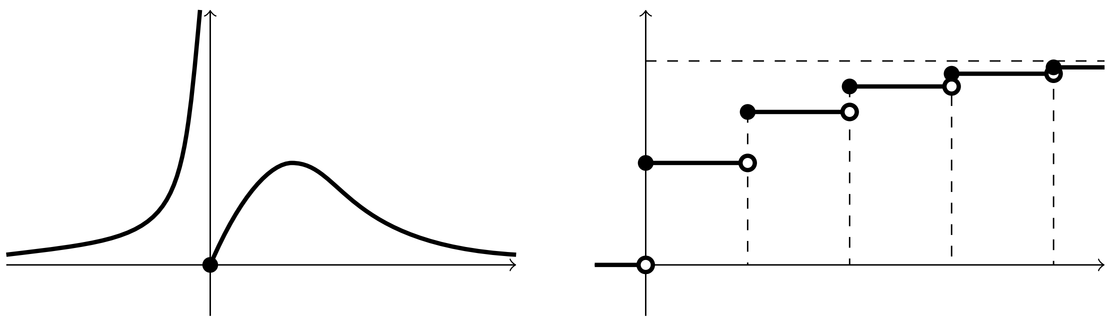



These two functions stay constant on the left and right.

[1]The trivial bound 1 doesn't count.



On the left side, the pattern goes on periodically.
On the right side, the function stays constant on the left and right.



The two functions converge on the left and right to the value they seem to converge to.

Exercise F.

1. A continuous random variable X has probability density function

$$p_X(x) = \begin{cases} C(1-x^2) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

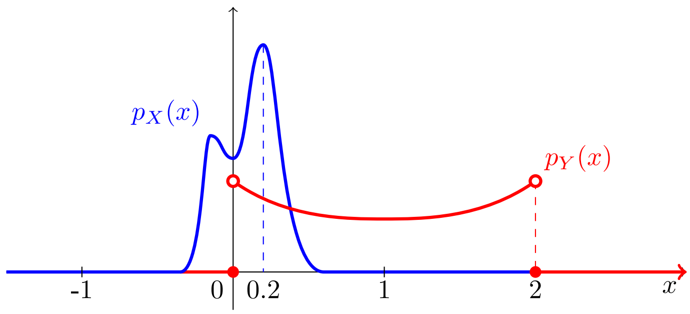
for some constant C. What is the value of C?

2. A continuous random variable Y has probability density function

$$p_Y(y) = \begin{cases} ay + b & \text{for } x \in (0,1), \\ 0 & \text{for } x \notin (0,1), \end{cases}$$

for some constants $a \geq 0$ and $b \in \mathbb{R}$. What are the possible values for a and b?

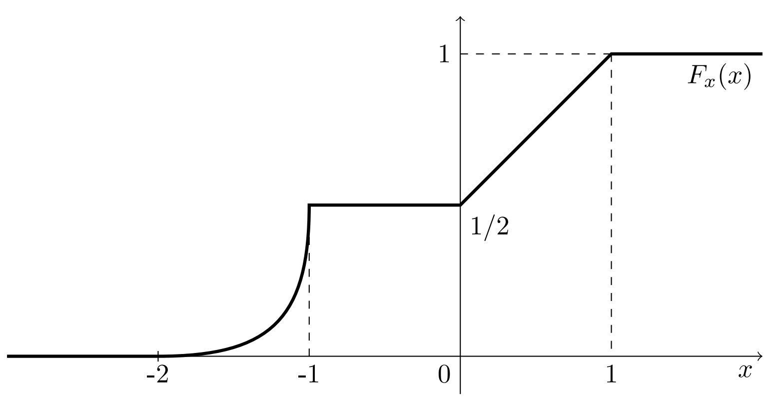
Exercise G. On the graph below are two densities associated to two variables X and Y.



Which of the following statements are true?

1. The mean of Y is larger than the mean of X.
2. The variance of X is larger than the variance of Y.
3. The median of X is positive.
4. The mean of Y is close to 1.
5. $\mathbb{P}(Y = 0)$ is positive.
6. It is more likely for X to be close to 0.2 than to 0.
7. It is more likely for Y than for X to fall in the interval $(0, 0.4)$.

Exercise H. Below is the graph of the cumulative distribution function of some continuous variable X. What is its median? its third quartile?



Usual continuous variables

Exercise I. A factory produces plastic pipes for plumbing. For each of the following variables X, give the name of its distribution and its parameter(s).

1. The average length of a standard tube is two metres. However, the cut does not have to be precise, and the machine makes a cut with a standard deviation of 2 centimetres.[2] X is the length of a tube in centimetres.
2. During the course of a day, the factory receives in average two calls from clients. X is the number of calls received from clients in a week.
3. The 24/7 hotline for plumbing emergencies receives calls only once a week in average. X is the number of days after January 1st before the hotline receives a call.
4. Pipes have technical information printed along the side, useful in case of damage. The label should always face the plumber (so it should be up when buried, for instance). A careless client simply lays them at random. X is the angle between the adequate and the actual positions.

Expectation/variance/moments of continuous variables

Exercise J.

1. A continuous random variable X has probability density function

$$p_X(x) = \begin{cases} C(1-x^2) & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

for some constant C, whose value is computed in Exercise F. What is the expectation $\mathbb{E}[\sin(X)]$?

2. A continuous random variable Y has probability density function

$$p_Y(y) = \begin{cases} \frac{2}{\ln 2} \cdot \frac{1}{y+y^3} & \text{for } y \geq 1, \\ 0 & \text{for } y < 1 \end{cases}$$

What is the expectation of Y?

Exercise K. We want a variable $X \sim B(a, b)$ to have expectation $2/3$, and to satisfy

$$\mathbb{P}(|X - 2/3| \geq 1/6) \leq 1/10.$$

Give an example of possible values for a and b.

Transformation of continuous variables

Exercise L.

1. Let X be a random variable with distribution $\mathcal{E}xp(2)$. What is the density of $\sqrt{X}$?
2. Let X be a random variable with distribution $\mathcal{U}nif([0, 1])$. What is the density of $X(1-X)$?
3. Let X and Y be independent random variables with distribution $\mathcal{U}nif([0, 1])$. What is the density of $\min(X, Y)$?

[2] 1m = 100cm.