Review 3

Continuous variables, expectation, variance

Exercise A.

1. X is a continuous random variable with density

$$p_X(x) = \begin{cases} 1+x & \text{for } x \in (-1,0) \\ 1-x & \text{for } x \in (0,1) \\ 0 & \text{else.} \end{cases}$$

   What is the cumulative distribution function of X? its expectation? its variance? its third quartile?

2. X is a continuous random variable with density

$$p_X(x) = \begin{cases} 2x^3 - \frac{3}{2}x + \frac{1}{2} & \text{for } x \in (-1,1) \\ 0 & \text{else.} \end{cases}$$

   Check that this is indeed a probability density function.

   What is the cumulative distribution function of X? its expectation? its variance?

Exercise B. Assume the lifetime T of a light bulb, measured in days, follows a distribution $\mathcal{E}xp(\ln(2)/\tau)$ (τ is the half-life of the light bulb). We screw the light bulb in place on a Sunday, midnight (so the first 24 hours of its life are a Monday).

1. Let N be the number of the day it dies; for instance, $N = 3$ if the light bulb dies on the first Wednesday. Find the probability mass function of N. Do you recognise its distribution?

2. What is the probability that the light bulb dies on a Sunday?

Exercise C. Let X be a continuous variable taking values in $[0, M]$ for some $M > 0$. Show that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\mathrm{d}x.$$

Hint: integrate by parts.

Transformations of continuous variables

Exercise D.

1. Let X be a variable with distribution $\mathcal{U}nif([-1,2])$. What is the density of $|X|$?
2. Let X be a continuous random variable with density
$$p_X(x) = \begin{cases} \cos(x) & \text{for } x \in \left(0, \frac{\pi}{2}\right) \\ 0 & \text{else.} \end{cases}$$
What is the density of $\tan(X)$? Give your answer without using trigonometric functions.

Exercise E. Let X and Y be two independent variables with distributions $\mathcal{E}xp(\lambda)$ and $\mathcal{E}xp(\mu)$. Find the density of $Z = \min(X, Y)$.

Do you recognise the distribution of Z?

Multivariate discrete variables

Exercise F. Let X and Y be two independent random variables, with respective distributions $\mathcal{B}in(n,p)$ and $\mathcal{U}nif(\{1,\dots,n\})$. Define $Z = X$ if $X \neq 0$, $Z = Y$ else.

What is the probability mass function of Z? its expectation?

Exercise G. Suppose you have two dice, one with 2 black and 4 white sides, and another with 4 black and 2 white sides. You choose one of them uniformly at random (for instance, toss a fair coin), then throw the chosen die twice. Let X be 1 if the first throw shows a black side, 0 if the side is white, and similarly for Y and the second throw.

1. What is the expectation of X? its variance?
2. Same question for Y. Are X and Y independent?

Exercise H. Let (a,b) be a starting point on the lattice $\mathbb{Z}^2$. Define (X,Y) as the position of a particle after one jump on one of the 4 closest neighbours, chosen uniformly.

1. What is the covariance of (X,Y)? Are X and Y independent?
2. Suppose instead that the particle has a probability one half to be lazy and stay at the same point; otherwise it has the same behaviour. What about the covariance now? Are they independent?

Exercise I. Let X and Y be two discrete random variables with integer values and joint probability mass function
$$p(x,y) = \begin{cases} \frac{e^{-1}}{(x+1)!} & \text{for } 0 \le y \le x, \\ 0 & \text{else.} \end{cases}$$

1. What is the probability mass function of the marginal X?
2. Compute the expectation
$$\mathbb{E}\left[\frac{2^X}{3^Y}\right].$$

Multivariate continuous variables

Exercise J. Recall that

$$\int_{-\infty}^{+\infty} \exp(-t^2)\mathrm{d}t = \sqrt{\pi}.$$

Let X and Y be continuous random variables with joint density

$$p(x,y) = C\exp(-y^2/2 + xy - x^2).$$

1. Find the constant C.
2. Find the density of the marginals X and Y.
3. Find the covariance of (X,Y).

Hint: Complete the square, and do one or two good change(s) of variables.

Exercise K. Let X and Y be continuous random variables with joint density

$$p(x,y) = \begin{cases} C\exp(-y) & \text{for } 0 \le x \le y, \\ 0 & \text{else.} \end{cases}$$

1. Find the constant C.
2. Find the density of the marginals X and Y.
3. Find the covariance of (X,Y).

Exercise L. Let $P = (X,Y)$ be a point uniformly distributed on the unit circle. In other words, (X,Y) is a continuous random vector with density

$$p(x,y) = \begin{cases} 1/A & \text{for } (x,y) \text{ in the unit cirle,} \\ 0 & \text{else} \end{cases}$$

for A the area of the unit circle.

What is the expectation of $\|P\|^2$?

Transformation of multivariate variables

Exercise M. Let X and Y be independent variables, uniform over $[-1,1]$. What is the density of $Z = XY$?

Exercise N. Let X and Y be independent random variables with distribution $\mathcal{N}(0,1)$. Set $A = X$ and $B = X + Y$.

1. Using no integrals, what is the covariance of (A,B)?
2. What is the density of (A,B)?

   Hint: Integrals involving $\exp(-t^2)$ for any type of t are difficult to compute. Leave them be until you can make them disappear.

Exercise O. Let $X_1, \ldots, X_n$ be n independent variables, uniform over $[0, 1]$.

1. What is the density of $\max(X_1, \ldots, X_n)$? Do you recognise this distribution?
2. What about $\min(X_1, \ldots, X_n)$?

Limit theorems

Exercise P. Let $X_1, \ldots, X_n$ be independent variables with distribution $\mathcal{N}(0, 1)$.

1. Using no integrals, show that
$$\mathrm{Var}(\sin(X_1)) \leq 1.$$
2. Assume you are given a random number generator (that is, access to the internet). How would you estimate
$$\int_{-\infty}^{+\infty} \sin(x) \exp\left(-x^2/2\right) \mathrm{d}x?$$
3. How confident would you be in your approximation? Compare, for $n \cdot \varepsilon^2 = 10$ (ε is the error you consent to make), the bounds given by Chebyshev's inequality and the central limit theorem.