Homework 1

February 12th
Due February 19th

Your assignment may be handwritten or typeset, but in any case it should be neat and readable. Your name and assignment number should be clearly visible (like on this document for example). It is due on Gradescope.

You are encouraged to work in groups for this assignment. However, the redaction should be done on one's own: do not copy some other student's work, or give your assignment to some other student. To consult textbooks or online resources is fair game; on the other hand, to look up the exact exercise and its solution is not. I will be available at my office hours[1] if you need help.

Exercises from the book

Included in this homework are exercises 1.1, 1.5, 1.18, 1.19, 1.20 and 1.22.

Exercise 1

We are interested in the number N involved in the following random experiment. We toss a coin and want to see two heads. How many throws do we have to perform for this to happen?

For instance, if a series of throws goes

heads, *tails*, *tails*, *tails*, **heads**, *tails*, **heads**, **heads**, *tails*...

then we have $N = 5$.

1. Describe two different possible sample spaces (you may keep the probability function to yourself), and how to recover N from the outcomes.
2. Describe the event $A = \{N \leq 3\}$ in your two spaces.
3. Suppose that a fellow student designed a third underlying sample space, and refuses to tell you what it is. Given $B = \{N = 4\}$, how would you recover $C = \{N \geq 5\}$ using only the events A and B?
4. Assuming the coin is fair, what are the probabilities of A, B and C?

[1] Monday 2:30-3:30, Thursday 10:30-11:30, or by appointment.

Exercise 2

In the U.S., about 57% of all households have (at least) a pet, 38% a dog, 25% a cat,[2] and 19% have a cat and a dog.[3] What is the probability that a uniformly chosen American household has a pet but no cat and no dog?

Exercise 3

This exercise is a bit more challenging. Partial answers are welcome.

Let $A_0, A_1, A_2, \ldots$ be any sequence of events in a probability space $(S, \mathbb{P})$. The goal of this exercise is to show that if $A \subset \bigcup_{n=0}^{\infty} A_n$, then

$$\mathbb{P}(A) \leq \sum_{n=0}^{\infty} \mathbb{P}(A_n).$$

If the sum is not convergent in the right hand side, we say that it is $+\infty$, and the inequality is obvious.

Define $B_0 = A_0$, $B_{n+1} = A_{n+1} \setminus \bigcup_{k=0}^{n} A_k$; for instance, the elements of B_2 are precisely the elements of A_2 that are neither in A_0 nor A_1.

1. Show that $\mathbb{P}(B_n) \leq \mathbb{P}(A_n)$ for all n. Deduce that $\sum_{n=0}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=0}^{\infty} \mathbb{P}(A_n)$.
2. Show that if $x \in A_n$ for some index n, then $x \in \bigcup_{k=0}^{n} B_k$. Deduce that $\mathbb{P}(A) \leq \mathbb{P}(\bigcup_{n=0}^{\infty} B_n)$.

   Hint: Consider, if it exists, the first k for which $x \in A_k$.
3. Show that the events $B_0, B_1, B_2, \ldots$ are pairwise disjoint.

   Hint: Build intuition on a drawing.
4. Use the previous questions to conclude.

[2] From the American Veterinary Medical Association, *Pet Ownership & Demographic*, 2018.
[3] Made up, loosely based on the above reference.