Quiz 1
Solution

March 3rd

1. What is the inclusion-exclusion formula?

   $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

2. A dear friend of yours has a daring sense of fashion. You've known her for a while now, and you know that on a good day, she has a 80% chance of wearing a monocle that matches her handbag. On a bad day, these odds go down to 40%. She is a rather positive person, and about 9 days out of 10 are good.

   You planned on meeting today, and you see her approaching, you see her outfit does not match. How worried should you be? In other words, what is (an expression for) the probability that today is a bad day?

   Set $G =$ "Good day", $M =$ "Matching outfit". Then we know

   $$\mathbb{P}(M|G) = 80\% = \frac{4}{5}, \qquad \mathbb{P}\big(M\big|G^{\complement}\big) = 40\% = \frac{2}{5}, \qquad \mathbb{P}(G) = \frac{9}{10}.$$

   We are looking for $\mathbb{P}(G^{\complement}|M^{\complement})$. Using Bayes' formula, and remembering that it often works hand in hand with the law of total probability, we get

   $$\begin{aligned}\mathbb{P}\big(G^{\complement}\big|M^{\complement}\big) &= \frac{\mathbb{P}\big(M^{\complement}\big|G^{\complement}\big)\mathbb{P}\big(G^{\complement}\big)}{\mathbb{P}\big(M^{\complement}\big)} = \frac{\mathbb{P}\big(M^{\complement}\big|G^{\complement}\big)\mathbb{P}\big(G^{\complement}\big)}{\mathbb{P}\big(M^{\complement}\big|G^{\complement}\big)\mathbb{P}\big(G^{\complement}\big) + \mathbb{P}\big(M^{\complement}\big|G\big)\mathbb{P}(G)} \\ &= \frac{\frac{3}{5}\cdot\frac{1}{10}}{\frac{3}{5}\cdot\frac{1}{10} + \frac{1}{5}\cdot\frac{9}{10}} = \frac{3}{3+9} = \frac{1}{4} = 25\%.\end{aligned}$$

   The good news is, there's probably no reason to worry. You should still check on your friends from time to time, though.

3. A spaceship radio has broken down and is sending out random sequences of 5 letters from time to time.

   (a) How many different messages can it send?

   This is 5 choices out of 26, ordered, with repetitions: 26^5.

   Ordered without repetitions would be $26^{\underline{5}}$; unordered without repetitions would be $\binom{26}{5}$.

   (b) Assuming the random sequence is uniform, what is (a formula for) the probability that a sequence contains the letters "SOS" in order?

   (For instance, "SOSST" contains "SOS", but "VSOAS" and "SJJSO" do not.)

   A naive way to describe a sequence containing "SOS" would be to give:

   - the position of "SOS": either SOS**, *SOS* or **SOS (3 possibilities);
   - the remaining two letters, ordered, possibly identical (26^2 possibilities).

   However, if we do so, we'll describe the sequence "SOSOS" twice! So the total number of such sequences is $3 \cdot 26^2 - 1$. Using question 3.(b), we can say that the probability under consideration is

$$\mathbb{P}(\text{The sequence contains “SOS”}) = \frac{3 \cdot 26^2 - 1}{26^5} \approx 0.02\%.$$