MATH 30530 P. Perruchaud
Introduction to Probability May 5th

Quiz 3
Solution

1. What is the density of the distribution $\mathcal{G}amma(\nu, \lambda)$?

$$p_X(x) = \begin{cases} \frac{\lambda^\nu}{\Gamma(\nu)} x^{\nu-1} \mathsf{e}^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. Let X be a continuous random variable with density

$$p_X(x) = \frac{\sin(x)^2}{\pi x^2},$$

and set $Y = \exp(X)$. What is the density of Y?

Since $Y > 0$, we have $p_Y(y) = 0$ for all $y \leq 0$. For any $y > 0$, we have

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\exp(X) \leq y) = \mathbb{P}(X \leq \ln(y)) = F_X(\ln(y)).$$

Taking the derivative, we get

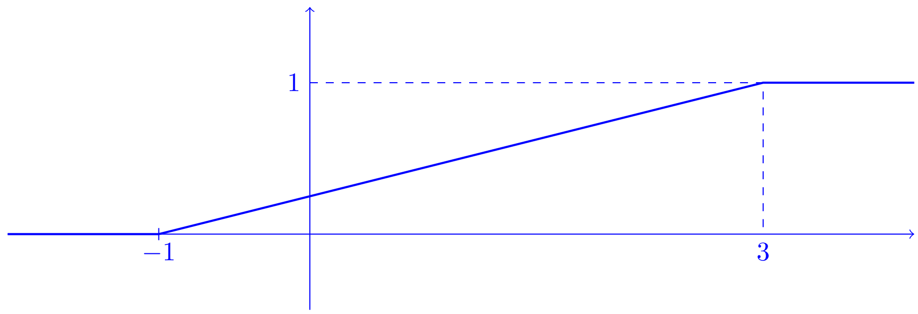
$$p_Y(y) = F_Y'(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_X(\ln(y)) = F_X'(\ln(y)) \cdot \frac{1}{y} = p_X(\ln(y)) \cdot \frac{1}{y} = \frac{\sin(\ln(y))^2}{y \cdot \pi \ln(y)^2}.$$

All in all,

$$p_Y(y) = \begin{cases} \frac{\sin(\ln(y))^2}{y \cdot \pi \ln(y)^2} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. Sketch the cumulative ditribution function of X, for X a continuous variable with distribution $\mathcal{U}nif(-1, 3)$.

The graph of the cumulative distribution function of a uniform variable is simple: zero before the lower bound, one after the upper bound, and a linear straight line joining the two. In our case, we get this graph.



If you which to get back to the formula for the density, we know that X can only take values between -1 and 3, so $F_X(x) = 0$ for $x \leq -1$ and $F_X(x) = 1$ for $x \geq 3$. For $-1 < x < 3$,

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} p_X(t)\mathrm{d}t = \int_{-1}^{x} \frac{1}{3-(-1)}\mathrm{d}t = \frac{t}{4}\Big|_{-1}^{x} = \frac{x+1}{4}.$$

This is indeed a straight line, with values 0 at -1 and 1 at 3.

4. Let X be a continuous random variable with density

$$p_X(x) = \begin{cases} \cos(2x) & \text{if } -\pi/4 < x < \pi/4, \\ 0 & \text{else.} \end{cases}$$

What is the expectation of X?

The traditional method goes as follows (note the integration by parts).

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{+\infty} x \cdot p_X(x)\mathrm{d}x \\ &= \int_{-\pi/4}^{\pi/4} x \cdot \cos(2x)\mathrm{d}x \\ &= \left(x \cdot \frac{\sin(2x)}{2}\right)\Big|_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4} \frac{\sin(2x)}{2}\mathrm{d}x \\ &= \left(\frac{\pi}{4} \cdot \frac{1}{2} - \frac{-\pi}{4} \cdot \frac{-1}{2}\right) - \left(\frac{-\cos(2x)}{4}\right)\Big|_{-\pi/4}^{\pi/4} \\ &= 0 - (-0 + 0) = 0. \end{aligned}$$

However, in this case, the density is even and we could have predicted the result. Indeed, setting $u = -x$,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot p_X(x)\mathrm{d}x = \int_{+\infty}^{-\infty} (-u) \cdot p_X(-u)(-\mathrm{d}u) = -\int_{-\infty}^{\infty} u \cdot p_X(u)\mathrm{d}u = -\mathbb{E}[X].$$

The only number equal to its opposite is zero, so $\mathbb{E}[X] = 0$.