

Review sheet

Preparation for Midterm 1

Elementary manipulations of sets

Exercise A. A customer buys ice cream at a parlour. Their choice is random, and the possibilities are as follows: the ice cream can be one or two scoops, and each scoop can be strawberry, chocolate or vanilla.

Write S and D for the event that the customer chooses one or two scoops ('single', 'double'), and B , C and V for the events that they choose at least one scoop of (straw)berry, chocolate or vanilla.

1. Write the event "The customer chooses only vanilla" as a formula using only the events described above.
2. Same question for the event "The customer chooses at most one strawberry scoop".

Exercise B. A baking show has three contestants initially: Andrei, Betty and Charlotte. During the first round, two contestants will pass the challenge, and one will lose and leave the game. During the second one, only one of the remaining two will pass and win the super baking crown.

Write A_1 (resp. B_1 , C_1) the event that Andrei (resp. Betty, Charlotte) passes the first round, and A_2 , B_2 , C_2 the events that they pass the second round. Give two examples of events that are (non-trivially) included,¹ and two examples of events that are disjoint.

Exercise C. We choose an American household at random. Denote by P , C , D , O the events that the household has at least one pet, at least one cat, at least one dog, and at least one pet that is not a cat nor a dog ('other').

1. Write the event "The household has at least one pet that is not a cat" as a formula using only the events described above.
2. Same question for the event "The household has at least one pet, but no cat".
3. Same question for the event "The household has one or many dogs, but nothing else".

¹Two events E and F are included one in the other if $E \subset F$ or $F \subset E$, and the inclusion is trivial if $E = F$.

Axioms of probability

Exercise D. Two events A and B are defined on some probability space. Which of the following probabilities are compatible with the axioms of probability? (There may be several good answers.)

1. $\mathbb{P}(A) = 50\%$, $\mathbb{P}(B) = 75\%$, $\mathbb{P}(A \cap B) = 20\%$.
2. $\mathbb{P}(A \cup B) = 70\%$, $\mathbb{P}(A) = 30\%$, $\mathbb{P}(B) = 30\%$.
3. $\mathbb{P}(A) = 50\%$, $\mathbb{P}(B) = 50\%$, $\mathbb{P}(A \cap B) = 0\%$.
4. $\mathbb{P}(A \cup B) = 75\%$, $\mathbb{P}(A) = 50\%$, $\mathbb{P}(A \cap B) = 50\%$.

Exercise E. You play They Love Me, They Love Me Not with a daisy: as you say the first ‘They love me’, you take off the first petal, then ‘They Love Me Not’ as you take off the second, and so on (so that they love you if the flower has 3 petals, for instance).

Daisies have between 8 to 12 petals, with probabilities²

$$\mathbb{P}(8 \text{ petals}) = \mathbb{P}(12 \text{ petals}) = \frac{1}{12}, \quad \mathbb{P}(9 \text{ petals}) = \mathbb{P}(11 \text{ petals}) = \frac{1}{4}, \quad \mathbb{P}(10 \text{ petals}) = p.$$

1. What should p be so that the probability space satisfies the axioms of probability?
2. What is the probability that your sweetheart loves you?

Exercise F. The probability that a randomly chosen American household has at least a cat is $\mathbb{P}(C) = 25\%$, at least a dog is $\mathbb{P}(D) = 38\%$, a TV is $\mathbb{P}(TV) = 96\%$, a computer is $\mathbb{P}(PC) = 90\%$.

1. What are the possible values for $\mathbb{P}(C \cup D)$?
2. What are the possible values for $\mathbb{P}(TV \cap PC)$?

Independence, conditioning

Exercise G. Two events A and B are defined on some probability space. Which of the following probabilities are compatible with the axioms of probability? (There may be several good answers.)

1. $\mathbb{P}(A) = 50\%$, $\mathbb{P}(B) = 50\%$, $\mathbb{P}(A|B) = 50\%$.
2. $\mathbb{P}(A) = 50\%$, $\mathbb{P}(B) = 20\%$, $\mathbb{P}(A|B) = 50\%$.
3. $\mathbb{P}(A) = 10\%$, $\mathbb{P}(B) = 20\%$, $\mathbb{P}(A|B) = 80\%$.
4. $\mathbb{P}(A) = 10\%$, $\mathbb{P}(B) = 50\%$, $\mathbb{P}(A|B) = 50\%$.

Exercise H. We throw three fair dice.

1. What is the probability $\mathbb{P}(\text{At least two } 1 \mid \text{The minimum is } 1)$?
2. What is the probability $\mathbb{P}(\text{At least one } 6 \mid \text{The minimum is } 1)$?

²Not biologically accurate.

Exercise I. We sample a random result X in the following fashion. We throw a fair die once. If it lands on a number from 1 to 5, the result X is the value of a second independent throw. If it lands on 6, we make a second independent throw, and the result X is 7 if the second throw is a number from 1 to 5, otherwise it is \star .

What are the probabilities of X being 1, 2, 3, 4, 5, 6, 7, \star ?

Given a biased coin that lands on heads with probability $0 < p < 1$, we sample a random result X in the following fashion. We toss the coin once first. The result X is the value of that first coin provided a second independent throw gives the opposite outcome. Otherwise, it is \star .

What are the probabilities of X being heads, tails, \star ?

(This last question is too opinion-based to be a midterm-type question.)

How would you emulate a fair 7-sided die with a single fair 6-sided die? How would you emulate a fair coin with a single unfair coin? You may throw them many times.

Law of total probability, Bayes law

Exercise J. 100 black-spotted white cows in a herd have random patterns around the eyes. We assume a black spot cannot cover both eyes. Which of the following collections of events are partitions? (There may be several good answers.)

1. B_1, \dots, B_{100} , where B_k is the event that the k th cow has a black spot around at least one eye.
2. B_0, \dots, B_{100} , where B_k is the event that at least k cows have a black spot around at least one eye.
3. B_0, \dots, B_{200} , where B_k is the event that there are exactly k black spots around the eyes of all the cows.
4. B_0, \dots, B_{100} , where B_k is the event that the k th cow is the first to have a black spot around at least one eye (B_0 is the event that none of them has such a spot).
5. B_0, \dots, B_{100} , where B_k is the event that all cows up to the k th one have a black spot on at least one eye, and the $(k+1)$ st doesn't (B_0 is the event that the first cow does not have such a spot).

Exercise K. A swimmer and a runner are competing in a television show involving two events: one on track and one in pool. The swimmer has a 40% chance to pass the track event, and a 60% chance to pass the pool event, independently from each other. The runner has the same probabilities, but the other way around.

We choose a first competitor uniformly at random, and watch them attempt the two trials.

1. Are the event 'The first competitor passes the track event' and 'The first competitor passes the pool event' independent?
2. What is the probability that the first competitor is the swimmer, knowing that they passed the pool trial?
3. What is the probability that the first competitor is the swimmer, knowing that they passed both trials?

Exercise L. Among the population of students being awarded a bachelor's degree in mathematics in the U.S. in a generation, we found the following statistics.³ About 8% of them went on to graduate with a PhD, and 5% of these remaining people were granted the title of professor.⁴ Out of all the graduates, about 43% were women; among the doctors, we find 29% of women, and it goes down to 15% for the professors.

1. What is the probability that a randomly chosen graduate raised to a professor position?
2. What is (a formula for) the probability that a randomly chosen graduate became a professor, knowing that she is a woman? Same question for a man.
3. Complete the following sentence:
'The social situation in academia makes male graduate in maths _____ times more likely to be awarded the title of professor than their female counterpart.'

Counting

Exercise M. Dumbledore, headmaster of the paleontology school Joedarts, organises each year the Fourhouse Tournament. He chooses 4 students at random out of the 400 in the school (100 for each house) to compete in various dinosaur-related tasks.

1. What is the probability that at least two students are chosen in the house Goldfirebird?

Dumbledore is notoriously biased towards Goldfirebird, and he will choose a set of four students uniformly out of all those that contain at least two students from this house.
2. What is the probability that all chosen students are from house Goldfirebird?
3. What is the probability that Dumbledore's favourite student Morty Parker (of course from house Goldfirebird) is chosen?

Exercise N. It seems like any three-letter word constructed in the form consonant-vowel-consonant is a valid English word. Here, we assume the English alphabet has 26 letters (from A to Z) and 5 vowels: A, E, I, O and U.

1. How many words of this form can we construct?
2. You are playing Scrabble against an opponent whose native language is not English, and you are fairly confident that you can convince him that any combination of this form is a genuine word. You have the letters B, H, L, T, V, A, O and you are the first to play, so nothing is on the board. How many such 'words' can you form?
3. Same question with the letters B, H, T, T, V, A and O.

³Based on data from the [National Science Foundation](#), and a few obviously false assumptions; in particular, the fact that the proportion of women is more or less constant.

⁴We are talking about actual professor, the highest rank in the U.S. academia. It is a very prestigious position.

Exercise O. You are dealt 5 cards of a standard deck of 52 cards in order.

1. What is the probability that you get all cards from the same suit?
2. What is the probability that you get all cards in strict increasing order?
3. What is the probability that you get at most two different ranks?

Exercise P. We throw a die 20 times.

1. What is the probability that exactly five of the throws are 2?
2. What is the probability to get only even numbers?
3. What is the probability to get at most three different numbers (with repetitions)?

Exercise Q. An online cooking videogame randomly assigns 9 players in different roles: 1 player is doing the dishes, 3 are cutting meat and vegetables, 3 are supervising the cooking, and 2 are taking food from place to place. The task partitioning is uniform among partitions satisfying the above requirements.

1. You and your buddy are playing that game. What is the probability that you end up in the same station?
2. Two groups of 4 and 2 people are playing and end up in the same game. What is the probability that for both teams, the players are all isolated?