Review sheet

Preparation for the final

Continuous random variables

Exercise A.

1. A continuous random variable X has probability density function

$$p_X(x) = \begin{cases} C(1-x^2) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

   for some constant C. What is the value of C?

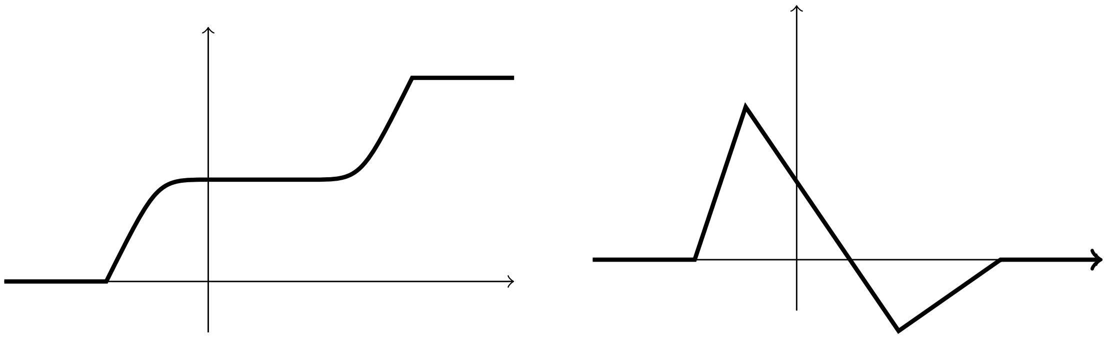
2. A continuous random variable Y has probability density function

$$p_Y(y) = \begin{cases} ay + b & \text{for } x \in (0,1), \\ 0 & \text{for } x \notin (0,1), \end{cases}$$

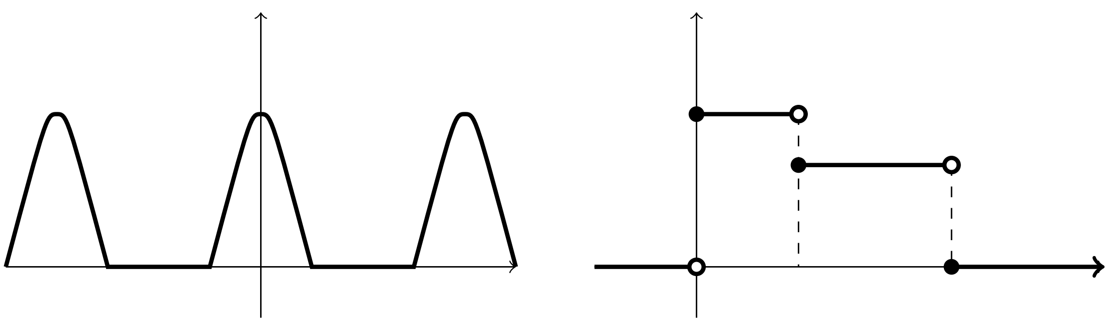
   for some constants $a \geq 0$ and $b \in \mathbb{R}$. What are the possible values for a and b?

Exercise B. Below are 6 graphs. For each of them,

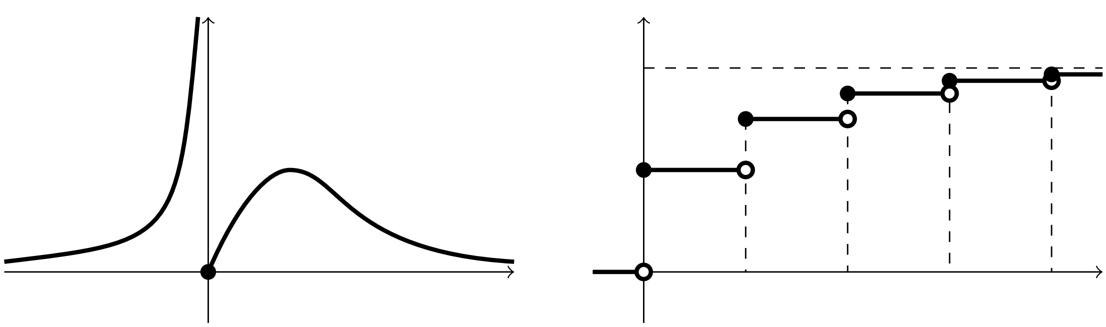
- determine if it is possible for the graph to be that of a cumulative distribution function or a density function;
- if it is the graph of a density, sketch the graph of the associated cumulative distribution function;
- if it is the graph of a cumulative distribution function, sketch the associated probability density function (if the variable is continuous) or probability mass function (if the variable is discrete).



These two functions stay constant to the left and right.

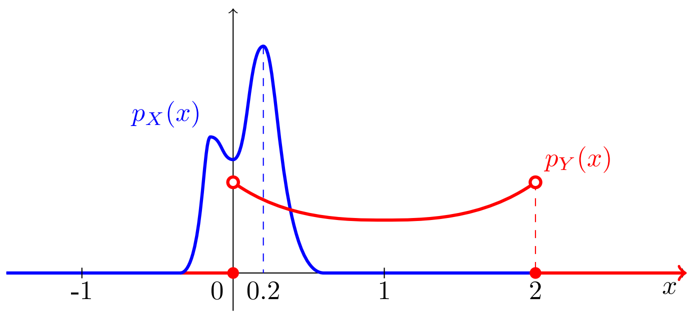


On the left side, the pattern goes on periodically.
On the right side, the function stays constant to the left and right.



These two functions converge on the left and right to the value they seem to converge to.

Exercise C. On the graph below are two densities associated to two variables X and Y.



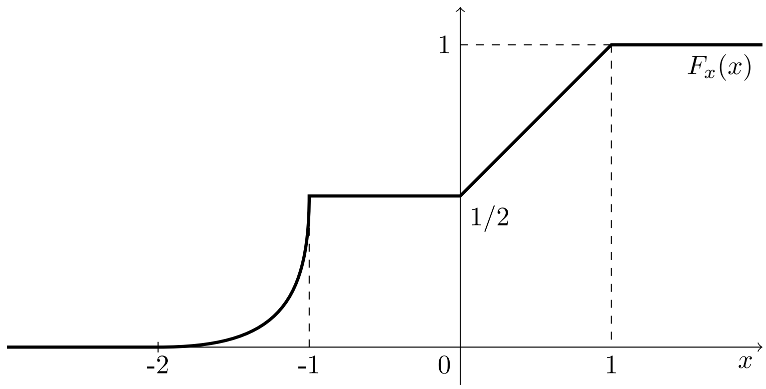
Which of the following statements seem to be true?

1. The mean of Y is larger than the mean of X.
2. The variance of X is larger than the variance of Y.
3. The median of X is positive.
4. The mean of Y is close to 1.
5. $\mathbb{P}(Y = 0)$ is positive.
6. It is more likely for X to be close to 0.2 than to 0.
7. It is more likely for Y than for X to fall in the interval $(0, 0.4)$.

Exercise D.

1. Draw the graph of the cumulative distribution function and the probability density function of a continuous random variable X such that
   - X takes only non-negative values;
   - X has at least a 20% chance of falling in the interval $[0, 1]$;
   - the median of X is at least 2;
   - X is half as likely to be close to $1/2$ than to 2.
2. Draw the graph of the probability density function of $1 + X/2$.

Exercise E. Below is the graph of the cumulative distribution function of some continuous variable X. What is its median? its third quartile?



Exercise F. Assume the lifetime T of a light bulb, measured in days, follows a distribution $\mathcal{E}xp(\ln(2)/\tau)$ (τ is the half-life of the light bulb). We screw the light bulb in place on a Sunday, midnight (so the first 24 hours of its life are a Monday).

1. Let N be the number of the day it dies; for instance, $N = 3$ if the light bulb dies on the first Wednesday (so T is continuous while N is, in a sense, its discrete counterpart). Find the probability mass function of N. Do you recognise its distribution?

2. What is the probability that the light bulb dies on a Sunday?

Usual continuous variables

Exercise G. A factory produces plastic pipes for plumbing. For each of the following variables X, give the name of its distribution and its parameter(s).

1. The average length of a standard tube is two metres. However, the cut does not have to be precise, and the machine makes a cut with a standard deviation of 2 centimetres.[1] X is the length of a tube in centimetres.

2. During the course of a day, the factory receives in average two calls from clients. X is the number of calls received from clients in a week.

3. The 24/7 hotline for plumbing emergencies receives calls only once a week in average. X is the number of days after January 1st before the hotline receives a call.

4. Pipes have technical information printed along the side, useful in case of damage. The label should always face the plumber (so it should be up when buried, for instance). A careless client simply lays them at random. X is the angle between the adequate and the actual positions.

Expectation/variance/moments of continuous variables

Exercise H.

[1] 1m = 100cm.

1. A continuous random variable X has probability density function

$$p_X(x) = \begin{cases} C(1-x^2) & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

for some constant C, whose value is computed in Exercise ???. What is the expectation $\mathbb{E}[\sin(X)]$?

2. A continuous random variable Y has probability density function

$$p_Y(y) = \begin{cases} \frac{2}{\ln 2} \cdot \frac{1}{y+y^3} & \text{for } y \geq 1, \\ 0 & \text{for } y < 1 \end{cases}$$

What is the expectation of Y?

Exercise I.

1. X is a continuous random variable with density

$$p_X(x) = \begin{cases} 1+x & \text{for } x \in (-1,0) \\ 1-x & \text{for } x \in (0,1) \\ 0 & \text{else.} \end{cases}$$

What is the cumulative distribution function of X? its expectation? its variance? its third quartile?

2. X is a continuous random variable with density

$$p_X(x) = \begin{cases} 2x^3 - \frac{3}{2}x + \frac{1}{2} & \text{for } x \in (-1,1) \\ 0 & \text{else.} \end{cases}$$

Check that this is indeed a probability density function.

What is the cumulative distribution function of X? its expectation? its variance?

Exercise J. We want a variable $X \sim B(a,b)$ to have expectation $2/3$, and to satisfy

$$\mathbb{P}(|X - 2/3| \geq 1/6) \leq 1/10.$$

Recall that for beta variables, we have

$$\mathbb{E}[X] = \frac{a}{a+b}, \qquad \operatorname{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Give an example of possible values for a and b.

Exercise K. Let X be a continuous variable taking values in $[0,M]$ for some $M > 0$. Show that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\mathrm{d}x.$$

Hint: integrate by parts.

Transformation of continuous variables

Exercise L. Let X be a continuous random variable with density

$$p_X(x) = \begin{cases} \cos(x) & \text{for } x \in \left(0, \frac{\pi}{2}\right) \\ 0 & \text{else.} \end{cases}$$

What is the density of $\tan(X)$? It is possible to give a final answer that doesn't involve trigonometric functions.

Exercise M.

1. Let X be a random variable with distribution $\mathcal{E}xp(2)$. What is the density of $\sqrt{X}$?
2. Let X be a random variable with distribution $\mathcal{U}nif([0,1])$. What is the density of $X(1-X)$?
3. Let X be a variable with distribution $\mathcal{U}nif([-1,2])$. What is the density of $|X|$?

Exercise N. Let X and Y be two independent variables with distributions $\mathcal{E}xp(\lambda)$ and $\mathcal{E}xp(\mu)$. What is the distribution of $Z = \min(X,Y)$?

Multivariate discrete variables

Exercise O. Let X and Y be two independent random variables, with respective distributions $\mathcal{B}in(n,p)$ and $\mathcal{U}nif(\{1,\ldots,n\})$. Define $Z = X$ if $X \neq 0$, $Z = Y$ else.

What is the probability mass function of Z? its expectation?

Exercise P. Suppose you have two dice, one with 2 black and 4 white sides, and another with 4 black and 2 white sides. You choose one of them uniformly at random (for instance, toss a fair coin), then throw the chosen die twice. Let X be 1 if the first throw shows a black side, 0 if the side is white, and similarly for Y and the second throw.

1. What is the expectation of X? its variance?
2. Same question for Y. Are X and Y independent?

Exercise Q. Let (a,b) be a starting point on the lattice $\mathbb{Z}^2$, i.e. the set of points in the plane with integer coordinates. Define (X,Y) as the position of a particle after one jump on one of the 4 closest neighbours, chosen uniformly.

1. What is the covariance of (X,Y)? Are X and Y independent?
2. Suppose instead that the particle has a probability one half to be lazy and stay at the same point; otherwise it has the same behaviour. What about the covariance now? Are they independent?

Exercise R. Let X and Y be two discrete random variables with integer values and joint probability mass function

$$p(x,y) = \begin{cases} \frac{\mathrm{e}^{-1}}{(x+1)!} & \text{for } 0 \leq y \leq x, \\ 0 & \text{else.} \end{cases}$$

1. What is the probability mass function of the marginal X?
2. Recall that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = \mathrm{e}^x.$$

   Compute the expectation

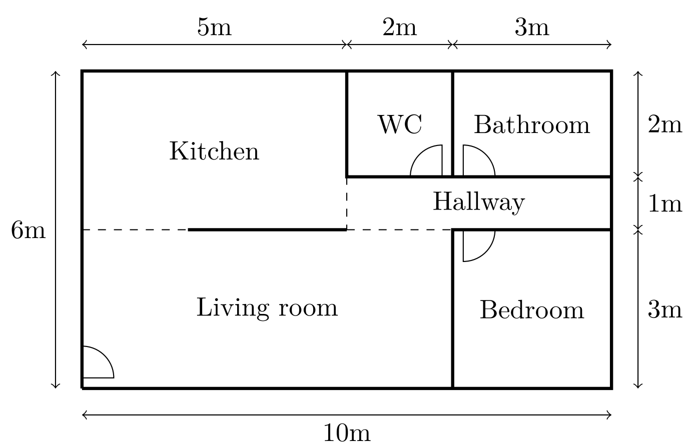
$$\mathbb{E}\left[\frac{2^X}{3^Y}\right].$$

Multivariate continuous variables

Exercise S. Let X and Y be two independent variables with distribution $\mathcal{E}xp(2)$.

1. What is the probability that $X + Y \leq 2$?
2. What is the density of $X + Y$? Do you recognise this distribution?

Exercise T. Below is the floor plan of Tina's house.



The position (X, Y) of her cat is a multivariate continuous random variable. Its density is zero outside the house, and it is twice as large in the living room than it is everywhere else in the house.

Tina likes her cat, but she also values her privacy. If she goes to the restroom, what is the probability that the cat is already there?

Exercise U. Let X and Y be continuous random variables with joint density

$$p(x,y) = \begin{cases} C\exp(-y) & \text{for } 0 \le x \le y, \\ 0 & \text{else.} \end{cases}$$

1. Find the constant C.
2. Find the density of the marginals X and Y.

Exercise V. Recall that

$$\int_{-\infty}^{+\infty} \exp(-t^2)\mathrm{d}t = \sqrt{\pi}.$$

Let X and Y be continuous random variables with joint density

$$p(x,y) = C\exp(-y^2/2 + xy - x^2).$$

1. Find the constant C.
2. Find the density of the marginals X and Y.

   Hint: Complete the square, and do one or two good change(s) of variables.

Exercise W. Let X and Y be continuous random variables with joint density

$$p(x,y) = \begin{cases} \frac{\exp(-x)}{x} & \text{for } 0 \le y \le x, \\ 0 & \text{else.} \end{cases}$$

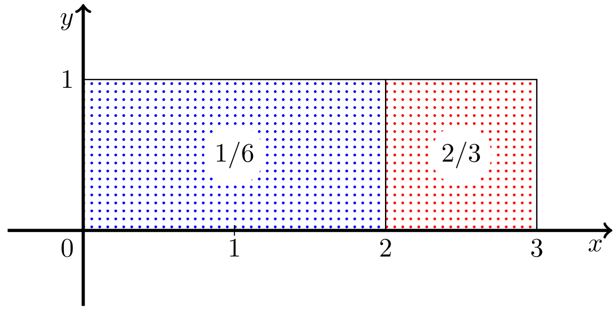
1. Find the probability that $X \le 1$.
2. Find the probability that $Y \le X/2$.

Exercise X. Let X and Y be independent variables, uniform over $[0,1]$. What is the density of $Z = XY$?

What about the case where X and Y are uniform over $[-1,1]$?

Exercise Y. Let X and Y be two variables which admit a joint probability density function p given by

$$p(x,y) = \begin{cases} 1/6 & \text{if } (x,y) \in (0,2) \times (0,1), \\ 2/3 & \text{if } (x,y) \in (2,3) \times (0,1), \\ 0 & \text{else.} \end{cases}$$



We imagine that X and Y are the price in a few months' time of some raw product. An investment company wants to choose between two strategies A and B, which have been designed to give a respective (random) profit e^{-X} and Y^2.

1. What are the probability density functions of the marginals X and Y?

2. What are the expectations of e^{-X} and Y^2? Which strategy is better in average?

3. The probability that the strategy A is better than the strategy B is equal to

$$\mathbb{P}\big((X,Y) \in R\big) = \iint_R p(x,y)\mathrm{d}x\mathrm{d}y$$

   for some region R of the plane. Draw R on a graph.

4. What is the probability that the strategy A is better than the strategy B? You might want to cut R into pieces over which the density is simple enough.