Homework 1
Solution

February 19th

Exercises from the book

Exercise 1.1. Set $a = \mathbb{P}(\text{Heads})$, $b = \mathbb{P}(\text{Tails})$, $c = \mathbb{P}(\text{Edge})$. Then

$$a + b + c = 1, \;\; c = 0.1, \;\; a = 2b.$$

We deduce

$$0.9 = a + b = 3b \Rightarrow b = 0.3, \;\; a = 0.6.$$

Exercise 1.5. Set A the event "rain on Saturday", and B the event "rain on Sunday". Then we are given the probabilities of, in order, A, B, $A \cap B$ and $A \cup B$.

(i) According to the inclusion-exclusion principle, we should have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.7 + 0.6 - 0.4 = 0.9.$$

This does not correspond to the given probability, so the given numbers are inaccurate.

(ii) These numbers correspond to the probability space $S = \{RR, RS, SR, SS\}$ ($R \leftrightarrow$ rainy, $S \leftrightarrow$ sunny) with the probability function satisfying

$$\mathbb{P}(RR) = 0.4, \quad \mathbb{P}(RS) = 0.3, \quad \mathbb{P}(SR) = 0.2, \quad \mathbb{P}(SS) = 0.1.$$

The sum of these probabilities is one, so it is a valid probability space.

(iii) We have $A \cap B \subset A$, so we must have

$$0.8 = \mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0.7.$$

This is obviously not the case, hence the given probabilities are inconsistent. The same argument also works for $A \cap B \subset B$, $A \subset A \cup B$, $B \subset A \cup B$, and most notably $A \cap B \subset A \cup B$.

(iv) Using the inclusion-exclusion principle again, we must have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.7 + 0.6 - 0.5 = 0.8.$$

This is again not satisfied, and these numbers do not correspond to any probability space.

Exercise 1.18 We have

$$\mathbb{P}(A|A\cup B) = \frac{\mathbb{P}(A\cap(A\cup B))}{\mathbb{P}(A\cup B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)+\mathbb{P}(B)}.$$

We used $A\cap(A\cup B) = A$; this type of set equality is proved by showing that an elements x belongs to the left hand side if and only if it belongs to the right hand side. In this case, it is not difficult.

Exercise 1.19 Because $B\cap C$ is not empty, C cannot be empty either. We have

$$\mathbb{P}(A\cap B\cap C) = \mathbb{P}(A\cap(B\cap C)) = \mathbb{P}(A|B\cap C)\mathbb{P}(B\cap C)$$

and

$$\mathbb{P}(B\cap C) = \mathbb{P}(B|C)\mathbb{P}(C),$$

and the first result follows directly.

As for the second identity,

$$\mathbb{P}(A|B\cap C) = \frac{\mathbb{P}(A\cap(B\cap C))}{\mathbb{P}(B\cap C)} = \frac{\mathbb{P}((A\cap B)\cap C)}{\mathbb{P}(B\cap C)} = \frac{\mathbb{P}(A\cap B|C)\mathbb{P}(C)}{\mathbb{P}(B|C)\mathbb{P}(C)} = \frac{\mathbb{P}(A\cap B|C)}{\mathbb{P}(B|C)}.$$

Exercise 1.20 Because A, B and C are independent, we have

$$\mathbb{P}(A\cap(B\cap C)) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(A)\mathbb{P}(B\cap C),$$

which is the definition of A and $B\cap C$ being independent.

Regarding A and $B\cup C$, note that

$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C);$$

indeed,

$$\begin{aligned} x\in A\cap(B\cup C) &\Leftrightarrow x \text{ belongs to } A\text{, and also to either } B \text{ or } C\\ &\Leftrightarrow x \text{ belongs to either } A \text{ and } B\text{, or } A \text{ and } C\\ &\Leftrightarrow x\in(A\cap B)\cup(A\cap C). \end{aligned}$$

(You may just write the equality without justification.)

Then, using the inclusion/exclusion principle, we get

$$\begin{aligned} \mathbb{P}\big(A\cap(B\cup C)\big) &= \mathbb{P}\big((A\cap B)\cup(A\cap C)\big) = \mathbb{P}(A\cap B)+\mathbb{P}(A\cap C)-\mathbb{P}(A\cap B\cap C)\\ &= \mathbb{P}(A)\mathbb{P}(B)+\mathbb{P}(A)\mathbb{P}(C)-\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(A)\big(\mathbb{P}(B)+\mathbb{P}(C)-\mathbb{P}(B)\mathbb{P}(C)\big)\\ &= \mathbb{P}(A)\big(\mathbb{P}(B)+\mathbb{P}(C)-\mathbb{P}(B\cap C)\big) = \mathbb{P}(A)\mathbb{P}(B\cup C), \end{aligned}$$

as expected.

Exercise 1.22 For $i = 1, 2$ denote by E_i the event "*the politician wins the i-th election*". We know that

$$\mathbb{P}(E_1) = 0.6, \qquad \mathbb{P}(E_2) = 0.5, \qquad \mathbb{P}(E_2|E_1) = 0.75.$$

(i)
$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_2|E_1)\mathbb{P}(E_1) = 0.75 \cdot 0.6 = 0.45.$$

(ii)
$$\mathbb{P}(E_1 \cap E_2^c) = \mathbb{P}(E_2^c|E_1)\mathbb{P}(E_1) = 0.25 \cdot 0.6 = 0.15.$$

(iii)
$$\mathbb{P}(E_1|E_2) = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_2)} = \frac{0.45}{0.5} = 0.9.$$

(iv)
$$\mathbb{P}(E_2|E_1^c) = \frac{\mathbb{P}(E_1^c|E_2)\mathbb{P}(E_2)}{\mathbb{P}(E_1^c)} = \frac{\big(1 - \mathbb{P}(E_1|E_2)\big)\mathbb{P}(E_2)}{\mathbb{P}(E_1^c)} = \frac{0.1 \cdot 0.5}{0.4} = 0.125.$$

Exercise 1

1, 2. A first possibility for S could be all the possible values for N: $\{2, 3, 4. \dots\} = \mathbb{N} \setminus \{0, 1\}$, where N is just the outcome. In this case, $A = \{2, 3\}$.

A second could be all the finite sequences of heads and tails with exactly two heads, and finishing with a heads:

$$S = \{(H, H), (H, T, H), (T, H, H), (H, T, T, H), (T, H, T, H), (T, T, H, H), \dots\}.$$

In this case, N is the length of the outcome, and $A = \{(H, H), (H, T, H), (T, H, H)\}$. The exercise did not ask for it, but the probability of a given outcome is $1/2^L = 2^{-L}$, where L is the length of the sequence.

A third one might be the set of couples (k, ℓ), where k is the number of tails we have to go through before the first heads, and ℓ the number of tails between the first and the second heads. In this case, N is given by $k + 1 + \ell + 1 = k + \ell + 2$. It means that $S = \mathbb{N}^2$, and the outcomes corresponding to the ones in the previous example are

$$S = \{(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \dots\}.$$

The probability of (k, ℓ) is $2^{-k-\ell-2}$, and the set A is $\{(0, 0), (1, 0), (0, 1)\}$ in this case.

I did cheat a bit in this third example: it is the second one in disguise, since there is a correspondence between their elements. An actual different example is the set of *unordered* pairs $\{k, \ell\}$, including $\{k, k\}$. Then $\{1, 3\}$ for instance may correspond to either (T, H, T, T, T, H) or (T, T, T, H, T, H), but it does not change N: is is still $k + \ell + 2$, independently of course of the ordering. The probability of $\{k, \ell\}$ is 2^{-k-l-2} if $k = \ell$, otherwise it is $2^{-k-\ell-1}$. In this instance, $A = \{\{0, 0\}, \{0, 1\}\}$.

A fifth and very natural possibility is the set of all sequences of heads and tails: $S = \{H, T\}^{\mathbb{N}}$. In this sample space, we have added useless tosses after the second heads, but of course we can recover N by finding the position of that second heads. The set A is the set of all sequences such that the first three elements are either (H, H, T), (H, H, H), (H, T, H) or (T, H, H). In this sample set, the probability of the event B consisting of all sequences $(x_0, x_1, x_2, \dots)$ such that elements $x_0, \dots, x_k$ (for instance, in our case we might want to consider $k = 3$) have fixed values $y_0, \dots, y_k$ (in our case, maybe $y_1 = y_2 = H$, $y_3 = T$) is $\mathbb{P}(B) = 2^{-k}$.

Many more sample spaces are available, although it is likely that those given above are the most convenient.

3. We have

$$C = \{N \geq 5\} = \{N < 4\}^{\complement} = \left(\{N \leq 3\} \cup \{N = 4\}\right)^{\complement} = (A \cup B)^{\complement}.$$

4. I will use the second probability space above.

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(\{(H,H),(H,T,H),(T,H,H)\}) \\ &= \mathbb{P}((H,H)) + \mathbb{P}((H,T,H)) + \mathbb{P}((T,H,H)) \\ &= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(\{(H,T,T,H),(T,H,T,H),(T,T,H,H)\}) \\ &= \mathbb{P}((H,T,T,H)) + \mathbb{P}((T,H,T,H)) + \mathbb{P}((T,T,H,H)) \\ &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}\end{aligned}$$

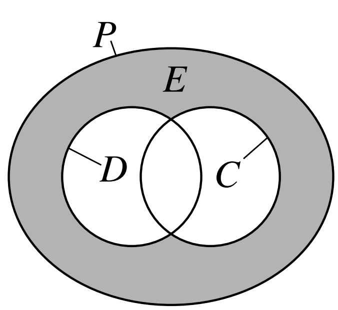
According to question 3, and noting that A and B are disjoint,

$$\mathbb{P}(C) = 1 - \mathbb{P}(A \cup B) = 1 - (\mathbb{P}(A) + \mathbb{P}(B)) = 1 - \left(\frac{1}{2} + \frac{3}{16}\right) = \frac{5}{16}.$$

Some of you may have used a variant of the fifth sample space above, and noted a sequence is in C if and only if its first 4 elements are either (T,T,T,T), (H,T,T,T), (T,H,T,T), (T,T,H,T) or (T,T,T,H), which gives the same result.

Exercise 2

We set P (resp. D, C) the probability that the randomly chosen household owns a pet (resp. a dog, a cat). Then the event we are interested in is $E = P \setminus (D \cup C)$.



Because $(D \cup C)$ is included in P, we have $P = \big(P \setminus (D \cup C)\big) \cup (D \cup C)$, and obviously the sets involved in this union are disjoint; hence

$$\mathbb{P}(P) = \mathbb{P}(E) + \mathbb{P}(D \cup C).$$

Using the inclusion-exclusion principle, we have

$$\mathbb{P}(D \cup C) = \mathbb{P}(D) + \mathbb{P}(C) - \mathbb{P}(D \cap C),$$

and finally

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(P) - \mathbb{P}(D \cup C) = \mathbb{P}(P) - (\mathbb{P}(D) + \mathbb{P}(C) - \mathbb{P}(D \cap C)) \\ &= 0.57 - 0.38 - 0.25 + 0.19 = 0.13.\end{aligned}$$

Exercise 3

The easiest way to prove that $\mathbb{P}(E) \leq \mathbb{P}(F)$ is to show that $E \subset F$. This exercise uses this trick over and over.

1. Of course $\mathbb{P}(B_0) \leq \mathbb{P}(A_0)$. Now by definition, B_{n+1} is A_{n+1} but with some elements taken away; it means that B_{n+1} is included in A_{n+1}, and $\mathbb{P}(B_{n+1}) \leq \mathbb{P}(A_{n+1})$. Of course it implies the second half immediately.

2. If $x \in A_n$, then in particular $x \in A_k$ for some k $(k = n)$. Choose k to be the smallest possible. If $k = 0$, then $x \in A_0 = B_0$ and obviously $x \in \bigcup_{k=0}^{n} B_k$.

Suppose now that $k > 0$. Then $x \notin A_0$, $x \notin A_1$,... , $x \notin A_{k-1}$, $x \in A_k$. In other words, x is not in any A_i, $i < k$, hence not in $\bigcup_{i=0}^{k-1} A_i$. Since x however belongs to A_k, x must belong to B_k by definition of B_k, and $x \in \bigcup_{k=0}^{n} B_k$ using the fact that $k \leq n$.

To show the second part, take x in A. We will show that $x \in \bigcup_{n=0}^{\infty} B_n$, hence $A \subset \bigcup_{n=0}^{\infty} B_n$ and the conclusion will follow.

Since x is in A, it is in $\bigcup_{n=0}^{\infty} A_n$. In particular, it must belong to some fixed A_m. But according to the first part of the question, we see that it is in fact in $\bigcup_{k=0}^{m} B_k$, hence in $\bigcup_{n=0}^{\infty} B_n$. This is indeed what we expected.

3. Let n and m be two different integers. We want to show that B_n and B_m are disjoint, i.e. that no elements belongs to the two of them.

Suppose first that $n < m$. We have seen in question 1 that B_n is included in A_n. But by definition, B_m is A_m where we have taken away all the elements from the previous A_k, including A_n. It means that every element of B_n has been removed to get B_m, so that their intersection is indeed empty.

If $m < n$, then it is just the same argument the other way around.

4. Using, in order, question 2, 3 and 1, we get

$$\mathbb{P}(A) \leq \mathbb{P}\Big(\bigcup_{n=0}^{\infty} B_n\Big) = \sum_{n=0}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=0}^{\infty} \mathbb{P}(A_n).$$