Homework 10
Solution

April 30th

Exercises from the book

Exercise 2.32 Since the density function $f(x)$ is odd we see that $\mathbb{E}[X] = 0$.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 e^{-x}\,dx$$

since $x^2 f(x)$ is even. Integrating by parts twice gives $\mathrm{Var}(X) = 2$. Alternatively, we can also notice that this last integral is the second moment of an exponential distribution of parameter 1; for $Y \sim \mathcal{E}xp(1)$,

$$\mathrm{Var}(X) = \int_0^{\infty} x^2 e^{-x}\,dx = \mathbb{E}[Y^2] = \mathrm{Var}(Y) + \mathbb{E}[Y]^2 = 1 + 1^2 = 2.$$

Exercise 2.34 Denote by X the random location where you break the stick, and by $L = L(X)$ the length of the longest of the two resulting segments. Then

$$L(X) = \begin{cases} 1 - X, & X \le \frac{1}{2}, \\ X, & X > \frac{1}{2}. \end{cases}$$

We have

$$\mathbb{E}[L] = \int_0^{\frac{1}{2}} (1-x)dx + \int_{\frac{1}{2}}^1 x dx = -\frac{(1-x)^2}{2}\Bigg|_0^{1/2} + \frac{x^2}{2}\Bigg|_{1/2}^1 = \frac{3}{4}.$$

Exercise 2.36 Suppose that the lifetime T of a light bulb is exponentially distributed, so $T \sim \mathcal{E}xp(\lambda)$. The probability that that a bult will last more that a year is $\mathbb{P}(T > 1) = e^{-\lambda}$ so we must have

$$0.8 \approx e^{-\lambda}.$$

The probability that a light bulb will last more than two years must be $e^{-2\lambda}$ so we must have

$$0.3 \approx e^{-2\lambda} = (e^{-\lambda})^2 \approx (0.8)^2.$$

This is clearly not the case so the lifetime is not exponentially distributed.

Exercise 2.37 The easy way is to not try and compute the cumulative distribution function of X nor Y. Since X cannot be negative, Y cannot either, so $p_Y(y) = 0$ for $y \leq 0$. We have

$$F_Y(y) = \mathbb{P}(\lambda X \leq y) = \mathbb{P}(X \leq y/\lambda) = F_X(y/\lambda).$$

Taking derivatives for $y > 0$, we see that

$$p_Y(y) = F_Y'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\Big(F_X(y/\lambda)\Big) = p_X\Big(\frac{y}{\lambda}\Big) \cdot \frac{1}{\lambda} = \mathsf{e}^{-y}.$$

Thus, $Y = \lambda X$ has the same cumulative distribution function as $\mathcal{E}xp(1)$.

If we wanted to follow the approach of the exercise, we would find $F_X(x) = 0$ for $x \leq 0$, and for $x > 0$

$$F_X(x) = \int_{-\infty}^{x} p_X(u)\mathrm{d}u = \int_0^x \lambda \mathsf{e}^{-\lambda u}\mathrm{d}u = 1 - \mathsf{e}^{-\lambda x}.$$

Because of the above, this would mean $F_Y(y) = F_X(y/\lambda) = 1 - \exp(-y)$, the derivative of which being what we expected indeed.

Exercise 2.45 Let $Y = X^2$. Set $F_Y(y) := \mathbb{P}(Y \leq y)$. Because $0 \leq Y \leq 1$, we know that $p_Y(y) = 0$ for $y \notin (0, 1)$. For $0 < y < 1$, we have

$$F_Y(y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

for F_X the cumulative distribution function of X. The probability density function of Y is $F_Y'(y)$, so

$$\begin{aligned} F_Y'(y) &= \frac{1}{2\sqrt{y}}\Big(F_X'(\sqrt{y}) + F_X'(-\sqrt{y})\Big) \\ &= \frac{1}{2\sqrt{y}}\Big(\frac{1}{1-(-1)} + \frac{1}{1-(-1)}\Big) \\ &= \frac{1}{2\sqrt{y}}. \end{aligned}$$

In particular, Y is in fact a beta distribution with parameters $(\frac{1}{2}, 1)$.

If we wanted to follow the approach of the exercise, we would find $F_X(x) = 0$ for $x \leq -1$, $F_X(x) = 1$ for $x \geq 1$, and for $0 < x < 1$ we have

$$F_X(x) = \int_{-\infty}^{x} p_X(u)\mathrm{d}u = \int_{-1}^{x} \frac{1}{2}\mathrm{d}u = \frac{x+1}{2}.$$

Because of the above, this would mean $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \sqrt{y}$, the derivative of which being what we expected indeed.

Exercise 1

Write Y for the highest bid of the other contestants, so $Y \sim \mathcal{U}nif([70, 130])$.

1. If the bid x is at most 70, the probability is zero, and it is one if x is at least 130. If $70 < x < 130$, the probability of winning is

$$\mathbb{P}(\text{winning}) = \mathbb{P}(Y < x) = \int_{-\infty}^{x} p_Y(y)\mathrm{d}y = \int_{70}^{x} \frac{1}{130-70}\mathrm{d}y = \frac{x-70}{60}.$$

All in all, we get

$$\mathbb{P}(\text{winning}) = \begin{cases} 0 & \text{for } x \leq 70, \\ \frac{x-70}{60} & \text{for } 70 < x < 130, \\ 1 & \text{for } x \geq 130. \end{cases}$$

2. Write X for the gain. For x fixed, X can take at most two values: 0 or $100 - x$. The expectation is then

$$\mathbb{E}[X] = 0 \cdot \mathbb{P}(\text{losing}) + (100-x) \cdot \mathbb{P}(\text{winning}) = \begin{cases} 0 & \text{for } x \leq 70 \\ \frac{(x-70)(100-x)}{60} & \text{for } 70 < x < 130 \\ 100-x & \text{for } x \geq 130. \end{cases}$$

3. For $x \geq 130$, the gain is negative. We have to study $(x-70)(100-x)$. For instance, we can see that

$$(x-70)(100-x) = 225 - (x-85)^2,$$

so that $(x-70)(100-x)$ is at most 225, and it the maximum is reached at $x = 85$. Since $100 - 225/60 = 385/4$ is positive, it is the maximal possible expectation, and the most profitable bid in average is 85.

Exercise 2

1. A function p is the probability density function of a continuous variable if and only if p is nonnegative and its integral is equal to one. So we need a and b to be nonnegative, and

$$1 = \int_{-\infty}^{+\infty} p_X(x)\mathrm{d}x = \int_{-1}^{1} a\mathrm{d}x + \int_{0}^{1} b\mathrm{d}x = a + b.$$

2. Let us compute the first two moments of X.

$$\mathbb{E}[X] = \int_{-1}^{0} ax\,\mathrm{d}x + \int_{0}^{1} bx\,\mathrm{d}x = \left.\frac{ax^2}{2}\right|_{-1}^{0} + \left.\frac{bx^2}{2}\right|_{0}^{1} = \frac{b-a}{2}$$

$$\mathbb{E}[X^2] = \int_{-1}^{0} ax^2\,\mathrm{d}x + \int_{0}^{1} bx^2\,\mathrm{d}x = \left.\frac{ax^3}{3}\right|_{-1}^{0} + \left.\frac{bx^3}{3}\right|_{0}^{1} = \frac{a+b}{3}$$

Hence, the variance of X is

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{4(a+b) - 3(b-a)^2}{12}.$$

Remember that $a + b = 1$, so the variance is minimal when $|b - a|$ is maximal, which means that (a, b) equals either $(0, 1)$ or $(1, 0)$.

Exercise 3 See Exercise 2.37 and Exercise 2.45 above.