Homework 11
Solution

May 7th

Exercises from the book

Exercise 2.35

(i) $T \sim \mathcal{E}xp(\lambda)$, $10 = \mu = \mathbb{E}[T] = \frac{1}{\lambda}$ so that $\lambda = 0.1$

$$\mathrm{Var}(T) = \mu^2 = 100.$$

(ii)

$$\mathbb{P}(T \leq 5) = 10 \int_0^5 e^{-0.1t} dt = 1 - e^{-0.5}.$$

(iii)

$$\mathbb{P}(T \leq 30 | T > 25) = 1 - \mathbb{P}(T > 30 | T > 25) = 1 - \mathbb{P}(T > 5) = \mathbb{P}(T \leq 5).$$

(iv)

$$\mathbb{P}(T > \mathbb{E}[T]) = \mathbb{P}(T > 10) = e^{-10\lambda} = e^{-1}.$$

Exercise 2.42 We compute the cumulative distribution for Y, and deduce its density by taking the derivative. It will help us to keep in mind that the inverse of the logarithm is the exponential.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\ln(X) \leq y) = \mathbb{P}(X \leq e^y) = F_X(e^y).$$

Since $F_X(x) = 1 - e^{-3x}$ for all $x > 0$, we get that

$$F_Y(y) = 1 - \exp(-3e^y)$$

and

$$p_Y(y) = 3e^y \cdot \exp(-3e^y) = 3\exp(-3e^y + y).$$

Exercise 2.46 Again, we first compute the cumulative distribution for Y. Setting $f(x)$ such that $Y = f(X)$, we see that for $0 < y < 1$,

$$f(x) \le y \Leftrightarrow (x \le y \text{ and } x \le 1) \text{ or } (1/x \le y \text{ and } x > 1) \Leftrightarrow x \le y \text{ or } x \ge 1/y.$$

Then for $0 < y < 1$ we have

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\{X \le y\} \cup \{X \ge 1/y\}) = \int_0^y e^{-x}\,dx + \int_{1/y}^{\infty} e^{-x}\,dx = 1 - e^{-y} + e^{-1/y}.$$

Differentiating, for $0 < y < 1$ the density function is $p_Y(y) = e^{-y} + \frac{1}{y^2}e^{-1/y}$. Since $f(x)$ has values in $[0,1]$ for $x \ge 0$, Y has values in $[0,1]$ and $F_y(y) = 0$ out of $(0,1)$.

Exercise 2.48

(i)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{c_n}^{\infty} \frac{c_n}{x^{n+1}}\,dx = \frac{1}{nc_n^{n-1}} = 1$$

since $f(x)$ is the density of a random variable. Therefore, if $n > 1$

$$c_n = \frac{1}{n^{1/(n-1)}}$$

while if $n = 1$, then c_n could be anything.

(ii)

$$\mathbb{E}[X] = \int_{c_n}^{\infty} \frac{c_n}{x^n}\,dx.$$

This diverges if $n = 1$ but for $n > 1$ gives

$$\frac{1}{n-1} \cdot n^{(n-2)/(n-1)} = \frac{n}{(n-1)n^{1/(n-1)}}.$$

(iii) The random variable Z_n has range $[\ln c_n, \infty)$. For z in this range we have

$$F_{Z_n}(z) = \mathbb{P}(Z_n \le z) = \mathbb{P}(\ln(X_n) \le z) = \mathbb{P}(X_n \le e^z) = \int_{c_n}^{e^z} \frac{c_n}{x^{n+1}}\,dx = \frac{1}{nc_n^{n-1}} - \frac{c_n}{ne^{nz}}.$$

The density is then the derivative of the function above:

$$p_{Z_n}(z) = \begin{cases} c_n e^{-nz} & \text{for } z \ge \ln(c_n), \\ 0 & \text{else.} \end{cases}$$

(iv)

$$\mathbb{E}[X_n^{m+1}] = \int_{c_n}^{\infty} x^{m+1}\frac{c_n}{x^{n+1}}\,dx = \int_{c_n}^{\infty} \frac{c_n}{x^{n-m}}\,dx.$$

This is convergent only of $m < n - 1$.

Exercise 3.5 The joint probability mass function of the random vector (X, Y) is

$$\mathbb{P}(X = b, Y = 2b) = \mathbb{P}(X = 2b, X = b) = \frac{1}{2}, \qquad \mathbb{P}(X = b, Y = b) = \mathbb{P}(X = 2b, Y = 2b) = 0.$$

This shows that X and Y have identical distributions

$$p_X(b) = p_Y(b) = \frac{1}{2} = p_X(2b) = p_Y(2b).$$

(i) The conclusion follows from the above equalities. Indeed, X, Y have identical distributions so

$$\mathbb{E}[X] = \mathbb{E}[Y] = \frac{1}{2}b + \frac{2b}{2} = \frac{3b}{2}.$$

(ii) Using the law of the subconscious statistician we deduce

$$\mathbb{E}\left[\frac{Y}{X}\right] = \sum_{x,y} \frac{y}{x}\mathbb{P}(X = x, Y = y) = \frac{1}{2}\frac{2b}{b} + \frac{1}{2}\frac{b}{2b} = 1 + \frac{1}{4} = \frac{5}{4}.$$

(iii) We have

$$\mathbb{E}[Z] = b\mathbb{P}(Z = b) + 2b\mathbb{P}(Z = 2b).$$

From the law of total probability we deduce

$$\mathbb{P}(Z = b) = \underbrace{\mathbb{P}(Z = b|X = b)}_{=1-p(b)}\mathbb{P}(X = b) + \underbrace{\mathbb{P}(Z = b|X = 2b)}_{=p(2b)}\mathbb{P}(X = 2b) = \frac{1}{2}\big(\,1 - p(b) + p(2b)\,\big).$$

Similarly

$$\mathbb{P}(Z = 2b) = \underbrace{\mathbb{P}(Z = 2b|X = b)}_{=p(b)}\mathbb{P}(X = b) + \underbrace{\mathbb{P}(Z = 2b|X = 2b)}_{=1-p(2b)}\mathbb{P}(X = 2b) = \frac{1}{2}\big(\,1 - p(2b) + p(b)\,\big).$$

Hence

$$\begin{aligned}\mathbb{E}[Z] &= \frac{b}{2}\Big(\big(\,1 - p(b) + p(2b)\,\big) + 2\big(\,1 - p(2b) + p(b)\,\big)\Big)\\ &= \frac{b}{2}\big(3 + p(b) - p(2b)\,\big) = \frac{3b}{2} + \frac{b}{2}\underbrace{\left(\frac{1}{1 + e^{2b}} - \frac{1}{1 + e^{4b}}\right)}_{>0} > \frac{3b}{2} = \mathbb{E}[X].\end{aligned}$$

Exercise 3.6 As suggested, call $X_1, \dots, X_4$ the variables defined by $X_i = 1$ if the $i$the component works, $X_i = 0$ otherwise. This means that the X_i are independent Bernoulli variables; more precisely, $X_1 \sim \mathcal{B}er(0.9)$, $X_2 \sim \mathcal{B}er(0.8)$, $X_3 \sim \mathcal{B}er(0.6)$, $X_4 \sim \mathcal{B}er(0.6)$.

(i)

$$\mathbb{E}[X] = \mathbb{E}[X_1 + \cdots + X_4] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_4] = 0.9 + 0.8 + 0.6 + 0.6 = 2.9$$

(ii) Because the variables are independent,

$$\mathrm{Var}(X) = \mathrm{Var}(X_1+\cdots+X_4) = \mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_4) = 0.9\cdot 0.1+0.8\cdot 0.2+0.6\cdot 0.4+0.6\cdot 0.4 = 0.73$$

(iii)

$$\mathbb{P}(X > 0) = 1 - \mathbb{P}(X = 0) = 1 - \mathbb{P}(X_1 = 0) \times \cdots \times \mathbb{P}(X_4 = 0) = 1 - 0.9 \cdot 0.8 \cdot 0.6 \cdot 0.6 = 0.7408.$$

(iv) As far as I know, there is no clever trick for this one as there was for the ones before.

$$\begin{aligned}\mathbb{P}(X = 1) &= \mathbb{P}(\text{The first component is the only one working})\\ &\qquad + \cdots + \mathbb{P}(\text{The fourth component is the only one working})\\ &= 0.9 \cdot 0.2 \cdot 0.4 \cdot 0.4 + 0.1 \cdot 0.8 \cdot 0.4 \cdot 0.4 + 0.1 \cdot 0.2 \cdot 0.6 \cdot 0.4 + 0.1 \cdot 0.2 \cdot 0.4 \cdot 0.6\\ &= 0.0512\end{aligned}$$

Exercise 1

Write Y for the maximum of X_1 to X_n. It has to be between 0 and 1, so we already know that $p_Y(y) = 0$ for $y \leq 0$ or $y \geq 1$. We choose $0 < y < 1$ and compute the cumulative distribution function.

$$\begin{aligned}F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(\max(X_1, \ldots, X_n) \leq y)\\ &= \mathbb{P}(X_1 \leq y \text{ and } \cdots \text{ and } X_n \leq y) = \mathbb{P}(X_1 \leq y) \times \cdots \times \mathbb{P}(X_n \leq y) = F_X(y)^n\end{aligned}$$

where $F_X(x)$ is the cumulative distribution function of one of the X_i (they all have the same one). Since they are uniform between 0 and 1, $F_X(x) = x$ for $0 < x < 1$, so for $0 < y < 1$ we have

$$F_Y(y) = y^n.$$

This means

$$p_Y(y) = \begin{cases} F'_Y(y) = ny^{n-1} & \text{for } 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

This is a Beta distribution $\mathcal{B}eta(n, 1)$.

Exercise 2

1. We must have $0 \leq X < Y$.

2. Fix x and y such that $0 \leq x < y$. Then

$$\begin{aligned}\mathbb{P}(X = x, Y = y) &= \mathbb{P}\big(\text{We do } y-1 \text{ throws giving 1,2,3,4 or 5, within which } x \text{ 1's, then a 6.}\big)\\ &= \frac{\#\{\text{possible positions for the 1's}\} \cdot \#\{\text{possible outcomes for the remaining throws}\}}{6^y}\\ &= \frac{\binom{y-1}{x} \cdot 4^{y-1-x}}{6^y}\end{aligned}$$