Homework 2
Solution

February 26th

Exercises from the book

Exercise 1.26 We have

$$\mathbb{P}(G|T)\mathbb{P}(T) = \mathbb{P}(G \cap T) = \mathbb{P}(T|G)\mathbb{P}(G).$$

Hence

$$\frac{\mathbb{P}(G|T)}{\mathbb{P}(T|G)} = \frac{\mathbb{P}(G)}{\mathbb{P}(T)}.$$

Thus

$$\mathbb{P}(G|T) = \mathbb{P}(T|G) \Leftrightarrow \mathbb{P}(G) = \mathbb{P}(T).$$

Exercise 1.27 Let U_k denote the event "we pick the kth urn" and G the event "the second ball we sample is green". Then

$$\begin{aligned}\mathbb{P}(G) &= \mathbb{P}(G|U_1)\underbrace{\mathbb{P}(U_1)}_{1/20} + \mathbb{P}(G|U_2)\underbrace{\mathbb{P}(U_2)}_{1/20} + \cdots + \mathbb{P}(G|U_{20})\underbrace{\mathbb{P}(U_{20})}_{1/20} \\ &= \frac{1}{20}\Big(\mathbb{P}(G|U_1) + \cdots + \mathbb{P}(G|U_{20})\Big).\end{aligned}$$

For $k = 1, 2, \ldots, n$, the kth urn contains $20 - k$ green balls and $k - 1$ red balls. Thus, when we take two samples without replacement from the kth urn the probability that the second ball is green is

$$\frac{(k-1)(20-k) + (20-k)(19-k)}{19 \cdot 18} = \frac{(20-k)18}{19 \cdot 18} = \frac{20-k}{19}.$$

Note that this is in accordance with the result we have seen in class, that sampling two balls and looking at the colour of the second, disregarding the first, will not change the probabilities when compared to just sampling one ball.

We deduce

$$\mathbb{P}(G) = \frac{1}{20} \cdot \frac{19 + 18 + \cdots + 1 + 0}{19} = \frac{\frac{19 \cdot 20}{2}}{20 \cdot 19} = \frac{1}{2}.$$

Exercise 1.29 (a) The two rolls are independent events. If the first roll is 1, then the largest number will be equal to the number we get at the second roll. The probability that this number is 3 is 1 in 6.

(b) The probability that the largest number is 3 is equal to the probability that the second roll gives a number ≤ 3. This probability is $\frac{1}{2}$.

Exercise 1.30 Write A'_k for the probability that the second die rolls on a k. We have $\mathbb{P}(A_k) = \frac{1}{6}$, hence

$$\mathbb{P}(B_n \cap A_k) = \mathbb{P}(A_k \cap A'_{n-k}) = (A'_{n-k}) \cdot \mathbb{P}(A_k) = \frac{1}{6}\mathbb{P}(A_{n-k}),$$

$$\mathbb{P}(B_n)\mathbb{P}(A_k) = \frac{1}{6}\mathbb{P}(B_n).$$

Thus, the events A_k and B_n are independent if

$$\mathbb{P}(A_{n-k}) = \mathbb{P}(B_n).$$

Observe that if $n \leq k$, then $\mathbb{P}(A_{n-k}) = 0$ so the events B_n and A_k are not independent. If $n > k$, then B_n is independent of A_k if

$$\mathbb{P}(B_n) = \mathbb{P}(A_{n-k}) = \frac{1}{6}.$$

This is possible only if $n = 7$. Thus A_k is independent of B_n if and only $n = 7$, for any $1 \leq k \leq 6$.

Exercise 1.34 Denote by N_k the event "the number k does not appear during 3 successive rolls of the die". We are interested in the probability $\mathbb{P}(N_5|N_6)$. We have

$$\mathbb{P}(N_5|N_6) = \frac{\mathbb{P}(N_5 \cap N_6)}{\mathbb{P}(N_6)}.$$

We have

$$\mathbb{P}(N_6) = \frac{5^3}{6^3}, \ \ \mathbb{P}(N_5 \cap N_6) = \frac{4^3}{6^3},$$

so

$$\mathbb{P}(N_5|N_6) = \frac{4^3}{5^3} = \frac{64}{125} = 0.512.$$

Exercise 1.36 Write A_i (resp. B_i, AB_i, O_i) for the event that the individual individual i has blood type A (resp. B, AB, O), and $\oplus_i$ (resp. $\ominus_i$) for the event that their Rh factor is positive (resp. negative). For the combined events, we will write for instance A_i^- for $A_i \cap \ominus_i$, and similarly for all blood types.

(i)

$$\mathbb{P}(A_1^- \cap A_2^-) = \mathbb{P}(A_1 \cap \ominus_1)^2 = \big(\mathbb{P}(A_1) \cdot \mathbb{P}(\ominus_1)\big)^2 = (0.4 \times 0.16)^2 \approx 0.0041.$$

(ii)

$$\mathbb{P}(O_i^+) = \mathbb{P}(O_i \cap \oplus_i) = 0.45 \cdot 0.84 = 0.378,$$

$$\mathbb{P}\Big((O_i^+)^{\complement} \Big) = 1 - 0.378 = 0.622.$$

The probability of the event described in the question is

$$\mathbb{P}\left(\left(O_1^+ \cap (O_2^+)^{\complement}\right) \cup \left((O_1^+)^{\complement} \cap O_2^+\right)\right).$$

Because this union is disjoint, the probability is in fact equal to

$$\begin{aligned}\mathbb{P}\left(O_1^+ \cap (O_2^+)^{\complement}\right) + \mathbb{P}\left((O_1^+)^{\complement} \cap O_2^+\right) &= 2\,\mathbb{P}(O_1^+) \cdot \mathbb{P}\Big((O_2^+)^{\complement}\Big) \\ &= 2 \cdot 0.378 \cdot 0.622 \\ &\approx 0.470.\end{aligned}$$

(iii)

$$\mathbb{P}(O_1^+ \cup O_2^+) = 1 - \mathbb{P}\left((O_1^+)^{\complement} \cap (O_2^+)^{\complement}\right) = 1 - 0.622^2 \approx 0.613.$$

(iv) Denote by E the event of interest: "one person is Rh positive and the other is not AB". Then

$$E = \underbrace{\left(\,\oplus_1 \cap (AB_2)^{\complement}\right)}_{E_1} \cup \underbrace{\left(\,\oplus_2 \cap (AB_1)^{\complement}\right)}_{E_2}.$$

Let us transform 'or' statements into 'and' statements (note that the Rh factors and groups of the individuals are 4 independent variables):

$$\begin{aligned}\mathbb{P}(E) &= 1 - \mathbb{P}\big(E_1^{\complement} \cap E^{\complement}\big) \\ &= 1 - \big(1 - \mathbb{P}(E_1)\big)(1 - \mathbb{P}(E_2)) \\ &= 1 - \Big(1 - \mathbb{P}(\oplus_1) \cdot \big(1 - \mathbb{P}(AB_2)\big)\Big)^2 \\ &= 2\,\mathbb{P}(\oplus_1)\big(1 - \mathbb{P}(AB_2)\big) - \Big(\mathbb{P}(\oplus_1)\big(1 - \mathbb{P}(AB_2)\big)\Big)^2 \\ &= 2(0.96 \cdot 0.84) - (0.96 \cdot 0.84)^2 \\ &\approx 0.9625.\end{aligned}$$

(v) The probability is

$$\mathbb{P}(A_1)^2 + \mathbb{P}(B_1)^2 + \mathbb{P}(O_1)^2 + \mathbb{P}(AB_1)^2 = (0.4)^2 + (0.11)^2 + (0.45)^2 + (0.04)^2 = 0.3762.$$

(vi) The event that they have different Rh factors (written $R_{\neq}$) is independent of the event that they have equal ABO type (written $ABO_=$, and whose probability we computed above). The probability must then be

$$\begin{aligned}\mathbb{P}(R_{\neq}) \cdot \mathbb{P}(ABO_=) &= \mathbb{P}\big((\oplus_1 \cap \ominus_2)) \cup (\ominus_1 \cap \ominus_2)\big) \cdot \mathbb{P}(ABO_=) \\ &= 2 \cdot \mathbb{P}(\oplus_1) \cdot \mathbb{P}(\ominus_2) \cdot \mathbb{P}(ABO_=) \\ &= 2 \cdot 0.84 \cdot 0.16 \cdot 0.3762 \\ &\approx 0.101.\end{aligned}$$

Exercise 1.39 Denote by $C \nrightarrow S$ the event "the luggage is missing in Sidney", by $C \nrightarrow LA$ the event "the luggage was lost between O'Hare and LAX" (mishandled in O'Hare) and by $LA \nrightarrow S$ the event "the luggage was lost between LAX and Sydney" (mishandled at LAX). We use $\rightarrow$ instead of $\nrightarrow$ for their complements. We know that for $p = 1\%$,

$$\mathbb{P}(C \nrightarrow LA) = p, \quad \mathbb{P}(LA \nrightarrow S | C \rightarrow LA) = p.$$

(i) We want to compute $\mathbb{P}(C \nrightarrow LA | C \nrightarrow S)$. We have

$$\mathbb{P}(C \nrightarrow LA | C \nrightarrow S) = \frac{\mathbb{P}(C \nrightarrow S | C \nrightarrow LA)\mathbb{P}(C \nrightarrow LA)}{\mathbb{P}(C \nrightarrow S)},$$

$$\mathbb{P}(C \nrightarrow S) = \mathbb{P}(C \nrightarrow S | C \nrightarrow LA)\mathbb{P}(C \nrightarrow LA) + \mathbb{P}(C \nrightarrow S | C \rightarrow LA)\mathbb{P}(C \rightarrow LA).$$

Note that $\mathbb{P}(C \nrightarrow S | C \nrightarrow LA) = 1$, $\mathbb{P}(C \nrightarrow S | C \rightarrow LA) = \mathbb{P}(LA \nrightarrow S | C \rightarrow LA) = p$. Hence

$$\mathbb{P}(C \nrightarrow S) = p + p(1-p),$$

$$\mathbb{P}(C \nrightarrow LA | C \nrightarrow S) = \frac{p}{p + p(1-p)} = \frac{1}{2-p} \approx 0.5025.$$

(ii) We have

$$1 = \mathbb{P}(LA \nrightarrow S | C \nrightarrow S) + \mathbb{P}(C \nrightarrow LA | C \nrightarrow S)$$

so that

$$\mathbb{P}(LA \nrightarrow S | C \nrightarrow S) = 1 - \frac{1}{2-p} \approx 0.4975.$$

Exercise 1

It is not possible for the dog to eat the couch right away. So the probability that your dog ate the couch is the sum of the probability that (1) it ate it as a second activity, or (2) it ate it as a third activity, but did not as a second activity (note that these events are disjoints). Moreover, if it ate the couch at some step, then it must have watched out the window just before; the total probability is in fact the sum of the probability that (1) it watched out the window right away then ate the couch directly after, and (2) it did some activity, then watched out the window, then ate the couch. There are only two activities that can both be a first activity and lead to watching out the window: having a nap, or playing. Hence (2) decomposes as (2a) have a nap then watch out the window then eat the couch, and (2b) play then watch out the window the eat the couch.

Let us write A_1, A_2 and A_3 for the first, second and third activities. What we just discussed rewrites as

$$\begin{aligned}\mathbb{P}(\text{eventually ate the couch}) = \; & \mathbb{P}(A_1 = \text{window} \cap A_2 = \text{couch}) \\ & + \mathbb{P}(A_1 = \text{nap} \cap A_2 = \text{window} \cap A_3 = \text{couch}) \\ & + \mathbb{P}(A_1 = \text{play} \cap A_2 = \text{window} \cap A_3 = \text{couch}).\end{aligned}$$

The first probability is

$$\mathbb{P}(A_1 = \text{window} \cap A_2 = \text{couch}) = \mathbb{P}(A_2 = \text{couch} \,|\, A_1 = \text{window})\mathbb{P}(A_1 = \text{window}) = \frac{1}{10} \cdot \frac{1}{3} = \frac{1}{30}.$$

Similarly,

$$\begin{aligned}
\mathbb{P}(A_1 = \text{nap} \,\cap\, A_2 = \text{window} \,\cap\, A_3 = \text{couch}) \\
&= \mathbb{P}(A_3 = \text{couch} \,|\, A_1 = \text{nap} \,\cap\, A_2 = \text{window}) \cdot \mathbb{P}(A_1 = \text{nap} \,\cap\, A_2 = \text{window}) \\
&= \mathbb{P}(A_3 = \text{couch} \,|\, A_2 = \text{window}) \cdot \mathbb{P}(A_2 = \text{window} \,|\, A_1 = \text{nap}) \cdot \mathbb{P}(A_1 = \text{nap}) \\
&= \frac{1}{10} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{90}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(A_1 = \text{play} \,\cap\, A_2 = \text{window} \,\cap\, A_3 = \text{couch}) \\
&= \mathbb{P}(A_3 = \text{couch} \,|\, A_1 = \text{play} \,\cap\, A_2 = \text{window}) \cdot \mathbb{P}(A_1 = \text{nap} \,\cap\, A_2 = \text{window}) \\
&= \mathbb{P}(A_3 = \text{couch} \,|\, A_2 = \text{window}) \cdot \mathbb{P}(A_2 = \text{window} \,|\, A_1 = \text{play}) \cdot \mathbb{P}(A_1 = \text{play}) \\
&= \frac{1}{10} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{90}
\end{aligned}$$

We find

$$\mathbb{P}(\text{eventually ate the couch}) = \frac{1}{30} + \frac{1}{90} + \frac{1}{90} = \frac{1}{18}.$$

Exercise 2

Write O_1^S for the event "OK is sent for the first chamber", E_2^R for "Error is received for the second chamber", and similarly for all other combinations.

1. Because the chambers and transmissions are independent, the probability is the square of the probability

   $$\mathbb{P}\big(O_1^R \big| O_1^S\big).$$

   The answer must then be

   $$\mathbb{P}\big(O_1^R \big| O_1^S\big)^2 = 0.99^2 = 0.9801.$$

2. (a)

   $$\mathbb{P}\big(O_1^S \cap E_2^S\big) = \mathbb{P}\big(O_1^S\big) \cdot \mathbb{P}\big(E_2^S\big) = \frac{1}{5} \cdot \frac{4}{5} = \frac{4}{25}.$$

   (b) This is a classic application of the Bayes law:

   $$\begin{aligned}
   \mathbb{P}\big(E_1^S \big| O_1^R\big) &= \frac{\mathbb{P}\big(O_1^R \big| E_1^S\big) \cdot \mathbb{P}\big(E_1^S\big)}{\mathbb{P}\big(O_1^R\big)} = \frac{\mathbb{P}\big(O_1^R \big| E_1^S\big) \cdot \mathbb{P}\big(E_1^S\big)}{\mathbb{P}\big(O_1^R \big| E_1^S\big) \cdot \mathbb{P}\big(E_1^S\big) + \mathbb{P}\big(O_1^R \big| O_1^S\big) \cdot \mathbb{P}\big(O_1^S\big)} \\
   &= \frac{0.01 \cdot 0.2}{0.01 \cdot 0.2 + 0.99 \cdot 0.8} = \frac{1}{397} \approx 0.265\%.
   \end{aligned}$$

   (c) We make full use of the independence of the two chambers:

   $$\begin{aligned}
   \mathbb{P}\big(E_1^S \cup E_2^S \big| O_1^R \cap O_2^R\big) &= 1 - \mathbb{P}\big(O_1^S \cap O_2^S \big| O_1^R \cap O_2^R\big) \\
   &= 1 - \mathbb{P}\big(O_1^S \big| O_1^R\big)^2 \\
   &= 1 - \Big(1 - \mathbb{P}\big(E_1^S \big| O_1^R\big)\Big)^2 \\
   &= 1 - \Big(1 - \frac{1}{397}\Big)^2 \approx 0.503\%.
   \end{aligned}$$