Homework 3
Solution

March 5th

Exercises from the book

Exercise 1.7 There are $26^3 \cdot 10^3$ possible plate numbers and

- $26^{\underline{3}} \cdot 10^3$ plates with no duplicate letters,
- $26^3 \cdot 10^{\underline{3}}$ plates with no duplicate digits,
- $26 \cdot 10^3$ plates with identical letters,
- $26^3 \cdot 5^3$ plates with odd digits.

The corresponding probabilities are

$$p_a = \frac{26^{\underline{3}} \cdot 10^3}{26^3 \cdot 10^3} \approx 0.8876, \qquad p_b = \frac{26^3 \cdot 10^{\underline{3}}}{26^3 \cdot 10^3} = 0.72,$$
$$p_c = \frac{26 \cdot 10^3}{26^3 \cdot 10^3} \approx 0.001479, \qquad p_d = \frac{26^3 \cdot 5^3}{26^3 \cdot 10^3} = 0.125.$$

Exercise 1.10 The probabilities we consider is the ratio of number of favourable outcomes by the number of possible outcomes. The outcomes are sequences of 3 distinct squares; if we consider them unordered, they represent $\binom{64}{3}$ possibilities.

(a) There are $\binom{8}{3}$ unordered sets of 3 squares in the first row, i.e. $\binom{8}{3}$ favourable outcomes for the event "all pieces are in the first row". The probability is

$$\mathbb{P}(\text{All pieces are in the first row}) = \frac{\binom{8}{3}}{\binom{64}{3}} \approx 0.00134.$$

Maybe, if the pieces are different for instance, we want to consider ordered sequences. In this case, we could follow a different approach that would give us the same result:

$$\mathbb{P}(\text{All pieces are in the first row}) = \frac{8^{\underline{3}}}{64^{\underline{3}}} \approx 0.00134.$$

(b) There are $\binom{32}{3}$ unordered sets of 3 black squares, i.e. $\binom{32}{3}$ favourable outcomes for the event "all pieces are on black squares". The probability is

$$\mathbb{P}(\text{All pieces are in the first row}) = \frac{\binom{32}{3}}{\binom{64}{3}} \approx 0.119.$$

Again, an ordered version would yield the same result:

$$\mathbb{P}(\text{All pieces are in the first row}) = \frac{32^{\underline{3}}}{64^{\underline{3}}} \approx 0.119.$$

(c) Because they are disjoint events, the probability that the pieces are all in the same row is the sum of the probabilities that they are in each individual row. Since this does not depend on the choice of the row, it means that the total probability is

$$\mathbb{P}(\text{All pieces are in the same row}) = 8 \cdot \mathbb{P}(\text{All pieces are in the first row}) = 8 \cdot \frac{\binom{8}{3}}{\binom{64}{3}} \approx 0.0107.$$

In the ordered case:

$$\mathbb{P}(\text{All pieces are in the same row}) = 8 \cdot \mathbb{P}(\text{All pieces are in the first row}) = 8 \cdot \frac{8^{\underline{3}}}{64^{\underline{3}}} \approx 0.0107.$$

(d) This is more or less the same reasoning. The difference is that we have to isolate not only a row, but also a colour. The total probability will be 8×2 times the probability of all pieces being on a fixed row and a fixed colour (say the black squares on the first row), and the number of outcomes corresponding to that event is $\binom{4}{3}$. All in all, we have

$$\begin{aligned}
&\mathbb{P}(\text{All pieces are in the same row and on the same colour}) \\
&\qquad = 8 \cdot 2 \cdot \mathbb{P}(\text{All pieces are in the first row on black squares}) \\
&\qquad = 8 \cdot 2 \frac{\binom{4}{3}}{\binom{64}{3}} \approx 0.00154.
\end{aligned}$$

In the ordered case:

$$\mathbb{P}(\text{All pieces are in the same row and on the same colour}) = 8 \cdot 2 \frac{4^{\underline{3}}}{64^{\underline{3}}} \approx 0.00154.$$

Exercise 1.14.

(a) If the smallest number is 4, then the other chosen numbers belong to $\{5, \ldots, 10\}$. The number of possible outcomes is $10^{\underline{3}}$. The favourable outcomes are of three types, depending on when we have chosen the number 4, at the first draw, at the second draw, or at the third draw. Each of these types has the same number of outcomes $6^{\underline{2}}$ so that the probability is

$$\frac{3 \cdot 6^{\underline{2}}}{10^{\underline{3}}} = \frac{3 \cdot 6 \cdot 5}{10 \cdot 9 \cdot 8} = \frac{1}{8} = 0.125.$$

(b) If the smallest number is 4 and the largest is 8, then the third chosen number can only be $5, 6, 7$. There are thus 3 choices for the intermediate number, which could have been chosen at one of the 3 draws. There are only 2 options for the remaining draws, $4, 8$ or $8, 4$. Hence

$$\frac{3 \cdot 3 \cdot 2}{10^{\underline{3}}} = \frac{6}{10 \cdot 9 \cdot 8} = \frac{1}{5 \cdot 8} = \frac{1}{40}.$$

(c) Arguing as in (b) we deduce that the probability is

$$\frac{(k-j-1) \cdot 3 \cdot 2}{n^{\underline{3}}}, \qquad k > j+1.$$

Exercise 1.15 If the first black ball comes at the k-th draw, then the first $(k-1)$ drawn balls where white. There are $n^{\underline{k-1}}$ ways of sampling without replacement $(k-1)$ white balls, and m different ways of choosing a black ball. The total number of possible outcomes is $n + m^{\underline{k}}$ so the probability of this event is

$$\frac{n^{\underline{k-1}} \cdot m}{(n+m)^{\underline{k}}}.$$

Exercise 1.16 Denote by E the event that no district was robbed more than once. We seek the probability of the complementary event E^{C}. Then

$$\mathbb{P}(E) = 1 - \mathbb{P}\big(E^{\mathsf{C}}\big).$$

When there are 6 districts and 6 roberries we have

$$\mathbb{P}\big(E^{\mathsf{C}}\big) = \frac{6^{\underline{6}}}{6^6}, \qquad \mathbb{P}(E) = 1 - \frac{6^{\underline{6}}}{6^6} \approx 98.46\%.$$

If there are 10 districts and 8 robberies, the answer is

$$1 - \frac{10^{\underline{8}}}{10^8}, \qquad \mathbb{P}(E) = 1 - \frac{10^{\underline{8}}}{10^8} \approx 98.19\%.$$

Exercise 1.40 Denote by D_k the event "we get the number k after one roll of the die", T_k the event "we get k tails in a row" and by T the event described in the exercise: "we get only tails when we first roll a die and then we flip a coin as many times as the number we get after the die roll".

Using the law of total probability we deduce

$$\mathbb{P}(T) = \sum_{k=1}^{6} \mathbb{P}(T|D_k)\mathbb{P}(D_k).$$

Observe that for any $k = 1, \ldots, 6$ we have

$$\mathbb{P}(T|D_k) = \mathbb{P}(T_k) = \frac{1}{2^k}, \;\; \mathbb{P}(D_k) = \frac{1}{6}.$$

Hence

$$\mathbb{P}(T) = \frac{1}{6}\sum_{k=1}^{6} 2^{-k} = \frac{1}{6}\left(1 - \frac{1}{2^6}\right) = \frac{21}{128}.$$

Exercise 1.42 Consider the events A "we get an ace when drawing a card" and S "we get a spade that is not an ace when drawing a card". Recall that the probability to see A before S is

$$\frac{\mathbb{P}(A)}{\mathbb{P}(A)+\mathbb{P}(S)}.$$

We see that the probability that we first get an ace is

$$\frac{\frac{4}{52}}{\frac{4}{52}+\frac{12}{52}}=\frac{4}{16}=\frac{1}{4}.$$

Exercise 1.46 Let E be the event "one ace was drawn from the thickened 2nd deck".

For $i=0,1,2$ we denote by A_i the event "there were i aces among the two cards drawn from the original first deck". Observe that

$$\mathbb{P}(A_0)=\frac{(48)_2}{(52)_2}=\frac{\binom{48}{2}}{\binom{52}{2}}=\frac{48\cdot 47}{52\cdot 51}\approx 0.8506,$$

$$\mathbb{P}(A_1)=\frac{2\cdot 4\cdot 48}{(52)_2}\approx 0.1447,$$

$$\mathbb{P}(A_2)=\frac{(4)_2}{(52)_2}\approx 0.0045.$$

We have

$$\mathbb{P}(A_0|E)=\frac{\mathbb{P}(E|A_0)\mathbb{P}(A_0)}{\mathbb{P}(E|A_0)\mathbb{P}(A_0)+\mathbb{P}(E|A_1)\mathbb{P}(A_1)+\mathbb{P}(E|A_2)\mathbb{P}(A_2)}.$$

The thickened deck has 54 cards and, given A_i, it has $4+i$ aces. Using this we deduce

$$\mathbb{P}(E|A_0)=\frac{4}{54},\ \ \mathbb{P}(E|A_1)=\frac{5}{54},\ \ \mathbb{P}(E|A_2)=\frac{6}{54}.$$

Hence

$$\begin{aligned}\mathbb{P}(A_0|E)&=\frac{\frac{4}{54}\mathbb{P}(A_0)}{\frac{4}{54}\mathbb{P}(A_0)+\frac{5}{54}\mathbb{P}(A_1)+\frac{6}{54}\mathbb{P}(A_2)}=\frac{4\mathbb{P}(A_0)}{4\mathbb{P}(A_0)+5\mathbb{P}(A_1)+6\mathbb{P}(A_2)}\\&=\frac{4\cdot 48\cdot 47}{4\cdot 48\cdot 47+5\cdot 2\cdot 4\cdot 48+6\cdot 4\cdot 3}=\frac{376}{459}\\&\approx 0.8192.\end{aligned}$$

Exercise 1.48 Denote by U_i the event "the i-th urn was picked" and by B_k the event "a ball labelled k was drawn".

(a) In this case $\mathbb{P}(U_i)=1/2$, in both cases $i=1,2$. We have to compute $\mathbb{P}(U_1|B_5)$. We have

$$\mathbb{P}(U_1|B_5)=\frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5)}=\frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)+\mathbb{P}(B_5|U_2)\mathbb{P}(U_2)}$$

$$=\frac{\frac{1}{10}\cdot 0.5}{\frac{1}{10}\cdot 0.5+\frac{1}{100}\cdot 0.5}=\frac{0.1}{0.11}=\frac{1}{11}\approx 0.909.$$

(b) In this case we have

$$\mathbb{P}(U_1) = \frac{10}{110}, \qquad \mathbb{P}(U_2) = \frac{100}{110}, \qquad \mathbb{P}(B_5) = \frac{2}{110}.$$

We deduce

$$\mathbb{P}(U_1|B_5) = \frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5)} = \frac{\frac{1}{10} \cdot \frac{10}{110}}{\frac{2}{110}} = \frac{1}{2}.$$

Exercise 1.52 (i) Denote by A the event that there are no accidents during the next successive n days after an accident. Then

$$\mathbb{P}(A) = (1 - p_1)(1 - p_2) \cdots (1 - p_n).$$

If you are so inclined, you can use the notation

$$\mathbb{P}(A) = \prod_{k=1}^{n} (1 - p_k).$$

(ii) Denote by B the event that there is exactly one accident during the next successive n days after an accident. It is the sum of

$$\mathbb{P}(\text{There is only one accident, on day } k) = \underbrace{(1 - p_1) \cdots (1 - p_{k-1})}_{\text{no accident before day } k} p_k \underbrace{(1 - p_1) \cdots (1 - p_{n-k})}_{\text{no accident after day } k}$$

for all k from 1 to 20. We have

$$\begin{aligned} \mathbb{P}(B) = {} & p_1(1 - p_1) \cdots (1 - p_{n-1}) + (1 - p_1)p_2(1 - p_1) \cdots (1 - p_{n-2}) \\ & + \cdots + (1 - p_1) \cdots (1 - p_{n-1})p_n. \end{aligned}$$