Homework 4 Solution

March 12th

Exercises from the book

Exercise 1.6. The password consists of 7 symbols, two of which are digits. There are

$$\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$$

choices of locations of the digits. For each such choice of locations, the number of possible combinations letters/digits is the same. Assume the alphabet has 26 letters. With repetitions, there are $26^5 \cdot 10^2$ possible passwords with 2 digits at a specified location. Thus, in this case, the number of possible passwords is

$$21 \cdot 26^5 \cdot 10^2 = 24,950,889,600.$$

Without repetitions, there are $26^{\underline{5}} \cdot 10^{\underline{2}}$ possible passwords with digits at a specified location so the total number of passwords is

$$21 \cdot \underbrace{(26 \cdot 25 \cdot 24 \cdot 23 \cdot 22)}_{26^{\underline{2}}} \cdot \underbrace{10 \cdot 9}_{10^{\underline{2}}} = 14,918,904,000.$$

Exercise 1.45 Denote $H =$ "coin toss yields Heads" and $F =$ "fake quarter".

(i) The answer is $\mathbb{P}(F) = \frac{1}{3}$.

(ii) Using Bayes' formula we deduce

$$\mathbb{P}(F|H) = \frac{\mathbb{P}(H|F)\mathbb{P}(F)}{\mathbb{P}(H)}$$

$$= \frac{1 \cdot \mathbb{P}(F)}{\mathbb{P}(H|F)\mathbb{P}(F) + \mathbb{P}\big(H|F^{\complement}\big)\mathbb{P}\big(F^{\complement}\big)} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3}} = \frac{1}{2}.$$

Exercise 1

Let the four different amounts of money placed in the 4 distinct envelopes be

$$a_1 < a_2 < a_3 < a_4,$$

A possible way to represent the outcomes is to write down the amounts in the order in which they appear; for instance, (a_3, a_2, a_1, a_4) and (a_1, a_2, a_4, a_3) are possible outcomes. For the first one, the strategy 2 and 3 are winning strategies, whereas strategy 1 makes us loose (we don't keep the maximal amount). For the second one, only strategy 3 makes us win.

Outcomes are ordered sequences of 4 distinct elements of $\{a_1, \dots, a_4\}$; this means there are $4^{\underline{4}} = 4!$ of them. They are assumed to be uniform, so the probability of some event A is the cardinal of A divided by 4!

- Strategy 1 makes us win if the first element of the outcome is a_4. Such an outcome is described entirely by the remaining 3 elements (ordered), none of which can be a_4: this represents $3^{\underline{3}} = 3!$ possibilities.

$$\mathbb{P}(\text{Strategy 1 wins}) = \frac{3!}{4!} = \frac{1}{4}$$

- Strategy 2 makes us lose if a_4 is in first position, and win if a_4 is in second position (a_4 will always be higher than the first amount). If a_4 is in fourth position, then we win only if a_3 was in first position (otherwise we will pick a_3 or lower after seeing the first envelope). If a_4 is in third position, then the second element has to be lower than the first for us to win, and a bit of brainstorming gives us the winning combinations:

$$(a_2, a_1, a_4, a_3), \quad (a_3, a_1, a_4, a_2), \quad (a_3, a_2, a_4, a_1).$$

  Letting A be the event of one of the three above draws happening, this means that the probability must be

$$\begin{aligned}\mathbb{P}(\text{Strategy 2 wins}) = {} & \mathbb{P}(a_4 \text{ is in the second envelope}) \\ & + \mathbb{P}(a_4 \text{ is in the fourth envelope and } a_3 \text{ in the first}) + \mathbb{P}(A).\end{aligned}$$

  The probability of A is $3/4!$. The number of outcomes giving a_4 in second position is the number of ways the remaining 3 amounts can be ordered: $3^{\underline{3}} = 3!$. The number of outcomes with a_4 and a_3 in positions 4 and 1 is the number of ways one can place a_1 and a_2 in the remaining two slots: $2^{\underline{2}} = 2$. Coming back to the formula, we get

$$\mathbb{P}(\text{Strategy 2 wins}) = \frac{3!}{4!} + \frac{2}{4!} + \frac{3}{4!} = \frac{11}{24}.$$

- Strategy 3 makes us lose if a_4 was in one of the first two envelopes. If it was in the third envelope, we win (it will always be higher than the first two). If it was in the last envelope, we lose only if a_3 was in the third envelope (otherwise it means that a_3 was in one of the first two, and the only one that is highest is a_4). So the probability is

$$\begin{aligned}\mathbb{P}(\text{Strategy 3 wins}) = {} & \mathbb{P}(a_4 \text{ is in the third envelope}) + \mathbb{P}(a_4 \text{ is in the fourth envelope}) \\ & - \mathbb{P}(a_4 \text{ is in the fourth envelope and } a_3 \text{ in the third}).\end{aligned}$$

Similarly to what was discussed above, there are $3^{\underline{3}} = 3!$ outcomes with a_4 in third (respectively in forth) position, because we have to choose the place for each remaining amount. The probability of a_4 and a_3 coming respectively third and fourth is $2^{\underline{2}} = 2$, corresponding to the possible ways to place a_1 and a_2. We deduce

$$\mathbb{P}(\text{Strategy 3 wins}) = \frac{3! + 3! - 2}{4!} = \frac{5}{12}.$$

Since

$$\frac{1}{4} < \frac{5}{12} < \frac{11}{24},$$

strategy 2 is the best, and strategy 1 the worst out of the proposed three.

Exercise 2

1. Let us try to understand what is needed to describe a two-pairs. We need to know where are the two pairs, which will be enough to know where the isolated card is. We need the rank of each pair, and of the isolated card. Finally, we have to identify the suits of each pair and of the isolated card.

There are $\binom{5}{2,2,1}$ ways to place the 5 cards of a hand into 3 categories of 2, 2 and 1 cards. However, there is a subtlety here; in doing so, there will be a first and a second pair; for instance, the partitions

$$(\{1,2\},\{3,4\},5), \qquad (\{3,4\},\{1,2\},5)$$

will be different.

There are many ways to deal with this issue. For instance, we could impose that the first pair will be the one with the highest rank. Depending on the approach, an other way that is likely to work is to simply divide the final product by two. I chose instead to count the number of unordered ways to divide a hand in two pairs and an isolated card: for instance, I consider

$$\{\{1,2\},\{3,4\},5\}, \qquad \{\{3,4\},\{1,2\},5\}$$

to be equal. Since I can group ordered such partitions by two according to what unordered partition they give me, there are exactly $\frac{1}{2}\binom{5}{2,2,1}$ ways to choose two pairs in an ordered hand of 5 cards, if we disregard the order of the pairs.

Once this is done, we can choose the ranks of the three groups involved, for instance from left to right in the hand. They must be distinct (otherwise we would have a four-of-a-kind or a full house), which represents $13^{\underline{3}}$ possibilities. Lastly, we choose the suits for each group. We have $4^{\underline{2}}$ possibilities for the first pair we encounter in the hand from the left, just as many for the other pair, and 4 possibilities for the isolated card.

In summary, the number of ordered two-pairs is

$$\underbrace{\frac{1}{2}\binom{5}{2,2,1}}_{\text{positions}} \cdot \overbrace{13^{\underline{3}}}^{\text{ranks}} \cdot \underbrace{4^{\underline{2}} \cdot 4^{\underline{2}} \cdot 4}_{\text{suits}}.$$

2. The probability that we are given a two-pairs is

$$\frac{\frac{1}{2}\binom{5}{2,2,1}\cdot 13^{\underline{3}}\cdot 4^{\underline{2}}\cdot 4^{\underline{2}}\cdot 4}{52^{\underline{5}}} = \frac{123,552\cdot 120}{2,598,960\cdot 120} = \frac{198}{4165} \simeq 0.048,$$

the same as the one in the book.

Exercise 3

1. We are looking for the probability of you being selected while Mike isn't.

Unordered approach. There are 20 choose 5 possible unordered groups of students. A group containing you but not Mike is entirely described by an unordered group of 4 people who are not you or Mike; the correspondence is given by adding or removing yourself from the group. Since there are 18 choose 4 such groups, the probability is

$$\mathbb{P}(\text{disappointment}) = \frac{\binom{18}{4}}{\binom{20}{5}} = \frac{5\cdot 15}{20\cdot 19} = \frac{15}{76} \approx 19.7\%.$$

Ordered approach. There are $20^{\underline{5}}$ possible ordered groups of students. A group containing you but not Mike is described by an ordered group of 4 people who are not you or Mike, together with your position in the group; the correspondence is given by adding or removing yourself from the group at the given position. Since there are $18^{\underline{4}}$ such groups and 5 positions within the group, the probability is

$$\mathbb{P}(\text{disappointment}) = \frac{5\cdot 18^{\underline{4}}}{20^{\underline{5}}} = \frac{5\cdot 15}{20\cdot 19}.$$

2. We are looking for the probability of you, Mike and Sarah being selected (call that event A) knowing that you and Mike have already been selected (call this one B). Because the space is uniform, the conditional probability is the size of $A \cap B$ (which is just A) divided by that of B.

Unordered approach. Similarly to the above, groups of five containing you and Mike correspond to groups of three out of the other students, so this gives 18 choose 3 possibilities. We can use the same reasoning for groups including Sarah, and we get

$$\mathbb{P}(A|B) = \frac{\binom{17}{2}}{\binom{18}{3}} = \frac{3}{18} = \frac{1}{6} \approx 16.6\%.$$

Ordered approach. Ordered groups of five containing you and Mike correspond to ordered groups of three out of the other students, together with the positions of you and Mike inside this group. This means $5^{\underline{2}}$ possibilities for your positions in the group (ordered, because we have to know which is which), and $18^{\underline{3}}$ possibilities for the remaining students. We can do something similar for groups including Sarah, which yields

$$\mathbb{P}(A|B) = \frac{5^{\underline{3}}\cdot 17^{\underline{2}}}{5^{\underline{2}}\cdot 18^{\underline{3}}} = \frac{3}{18}.$$