# Homework 5
# Solution

March 19th

## Exercises from the book

**Exercise 2.1** The range of $X$ is $\{1, 3\}$ ($x$ coordinates of the jumps), with

$$\mathbb{P}(X = 1) = \frac{1}{3}, \qquad \mathbb{P}(X = 3) = \frac{2}{3}$$

(heights of the jumps). Everything follows easily from this:

$$F(2) = \frac{1}{3}, \qquad \mathbb{P}(X > 1) = \frac{2}{3}, \qquad \mathbb{P}(X = 2) = 0, \qquad \mathbb{P}(X = 3) = \frac{2}{3}.$$

**Exercise 2.2** There are 36 possible outcomes when we roll a pair of dice. Six of them are doubles $(1, 1), \ldots, (6, 6)$, while the other thirty involve pairs of distinct numbers.

(a) We have

$$\mathbb{P}(X = 1) = \frac{1 + 2 \cdot 5}{36} = \frac{11}{36}, \qquad \mathbb{P}(X = 2) = \frac{1 + 2 \cdot 4}{36} = \frac{9}{36},$$
$$\mathbb{P}(X = 3) = \frac{1 + 2 \cdot 3}{36} = \frac{7}{36}, \qquad \mathbb{P}(X = 4) = \frac{1 + 2 \cdot 2}{36} = \frac{5}{36},$$
$$\mathbb{P}(X = 5) = \frac{1 + 2 \cdot 1}{36} = \frac{1}{12}, \qquad \mathbb{P}(X = 6) = \frac{1}{36}.$$

(b) We have

$$\mathbb{P}(X = 5) = \frac{2}{36} = \frac{1}{18}, \qquad \mathbb{P}(X = 4) = \frac{2 \cdot 2}{36} = \frac{1}{9}, \qquad \mathbb{P}(X = 3) = \frac{2 \cdot 3}{36} = \frac{1}{6},$$
$$\mathbb{P}(X = 2) = \frac{2 \cdot 4}{36} = \frac{2}{9}, \qquad \mathbb{P}(X = 1) = \frac{2 \cdot 5}{36} = \frac{5}{18}, \qquad \mathbb{P}(X = 0) = \frac{6}{36} = \frac{1}{6}.$$

**Exercise 2.3** (a) We must have

$$1 = \sum_{k=0}^{\infty} \frac{c}{2^k} = c \sum_{k=0}^{\infty} \frac{1}{2^k} = 2c \Rightarrow c = \frac{1}{2}.$$

(b) We have

$$\mathbb{P}(X > 0) = 1 - \mathbb{P}(X = 0) = \frac{1}{2}.$$

(c) The probability that $X$ is even is

$$\begin{aligned} &\mathbb{P}(X = 0) + \mathbb{P}(X = 2) + \mathbb{P}(X = 4) + \cdots \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{2} \cdot \frac{1}{2^4} + \cdots = \frac{1}{2} \left( 1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots \right) \\ &= \frac{1}{2} \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots \right) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}. \end{aligned}$$

**Exercise 2.4** (a) The range of $X$ is $\{0, 1, 2, 3, 4, 5\}$, and

$$\mathbb{P}(X = k) = \binom{5}{k} \left( \frac{48}{52} \right)^{5-k} \left( \frac{4}{52} \right)^{k}.$$

Later, we will call this distribution the binomial $\mathcal{B}in(5, 4/52)$.
(b) The range of $X$ is $\{0, 1, 2, 3, 4\}$ and

$$\mathbb{P}(X = 0) = \frac{\binom{48}{5}}{\binom{52}{5}}, \qquad \mathbb{P}(X = 1) = \frac{\binom{4}{1} \cdot \binom{48}{4}}{\binom{52}{5}}, \qquad \mathbb{P}(X = 2) = \frac{\binom{4}{2} \cdot \binom{48}{3}}{\binom{52}{5}}$$

$$\mathbb{P}(X = 3) = \frac{\binom{4}{3} \cdot \binom{48}{2}}{\binom{52}{5}}, \qquad \mathbb{P}(X = 4) = \frac{\binom{4}{4} \cdot \binom{48}{1}}{\binom{52}{5}}.$$

Later, we will call this distribution the hypergeometric $\mathcal{HG}eom(4, 48, 5)$.

**Exercise 2.5** The range of $X$ is $\{-2, 0, 2\}$. We have

$$\mathbb{P}(X = 2) = \mathbb{P}(X = -2) = \frac{1}{4}, \qquad \mathbb{P}(X = 0) = \frac{1}{2}, \qquad \mathbb{P}(X = x) = 0 \text{ otherwise.}$$

It means that the cumulative distribution function has 3 jumps at $-2$, 0 and 2, with sizes 1/4, 1/2 and 1/4: