Homework 6
Solution

March 26th

Exercises from the book

Exercise 2.6 Write a for the number you pick. Denote by X the number of a-s when we toss three dice, and by W your win. The range of X is $\{0, 1, 2, 3\}$. Denote by p the probability mass function of X. We have

$$W = \begin{cases} -1 & \text{if } X = 0, \\ X & \text{if } X > 0, \end{cases}$$

so

$$\mathbb{E}[W] = -p(0) + p(1) + 2p(2) + 3p(3).$$

Note that $X \sim \mathcal{B}in(3, 1/6)$. We have

$$p(1) + 2p(2) + 3p(3) = \mathbb{E}[X] = 3 \cdot \frac{1}{6} = \frac{1}{2}, \qquad p(0) = \mathbb{P}(X = 0) = \left(\frac{5}{6}\right)^3.$$

Hence

$$\mathbb{E}[W] = \frac{1}{2} - \left(\frac{5}{6}\right)^3 \approx -0.079.$$

Exercise 2.8 The number B of different birthdays is a random variable with range $\{1,2,3,4\}$. The pmf is computed as follows.

$$\mathbb{P}(B=1) = \underbrace{\frac{1}{365^4}}_{\text{number of possible birthdays}} \cdot \overbrace{365}^{\text{common birthday}} = \frac{1}{365^3},$$

$$\mathbb{P}(B=2) = \frac{1}{365^4} \overbrace{\binom{365}{2}}^{\text{two different birthdays}} \cdot \left[\underbrace{\binom{4}{3} + \binom{4}{2} + \binom{4}{1}}_{\text{3, 2 or 1 people for the first birthday}} \right] = \frac{364 \cdot 7}{365^3}$$

$$\mathbb{P}(B=3) = \frac{1}{365^4} \binom{365}{3} \cdot \overbrace{3}^{\text{which birthday is shared}} \cdot \underbrace{(4)_2}_{\text{2 isolated people}} = \frac{(364)_2 \cdot 6}{365^3}$$

$$\mathbb{P}(B=4) = \frac{(365)_4}{365^4} = \frac{(364)_3}{365^3}.$$

Using this data we compute the expectation to be

$$\mathbb{E}[B] = 1 \cdot \mathbb{P}(B=1) + 2 \cdot \mathbb{P}(B=2) + 3 \cdot \mathbb{P}(B=3) + 4 \cdot \mathbb{P}(B=4) \approx 3.98.$$

Exercise 2.9 Denote by H the number of heads Bob gets, and by G its gains. Then $H \in \{0,1,2,3\}$ and

$$G = \begin{cases} 0.25 \cdot H, & H > 0, \\ -2, & H = 0. \end{cases}$$

Thus

$$\begin{aligned} \mathbb{E}[G] &= 0.25 \cdot \mathbb{P}(H=1) + 0.5 \cdot \mathbb{P}(H=2) + 0.75 \cdot \mathbb{P}(H=3) - 2 \cdot \mathbb{P}(H=0) \\ &= 0.25 \cdot \frac{\binom{3}{1} + 2\binom{3}{2} + 3\binom{3}{3}}{2^3} - \frac{2}{2^3} = 0.25 \cdot \frac{3}{2} - \frac{1}{4} = \frac{1}{8}. \end{aligned}$$

$$\begin{aligned} \mathbb{E}[G^2] &= \frac{1}{16} \cdot \frac{\binom{3}{1} + 4 \cdot \binom{3}{2} + 9\binom{3}{3}}{8} + 4 \cdot \frac{1}{8} \\ &= \frac{3 + 12 + 9}{128} + \frac{64}{128} = \frac{88}{128} = \frac{11}{16}. \end{aligned}$$

$$\operatorname{var}(G) = \mathbb{E}[G^2] - \mathbb{E}[G]^2 = \frac{11}{16} - \frac{1}{64} = \frac{43}{64}.$$

Exercise 2.10 Note first that the order in which extract the balls is irrelevant when deciding which has the smallest label so we assume that we draw three balls simultaneously. There are $\binom{10}{3} = 120$ possibilities.

Next observe that

$$\mathbb{P}(X=10) = \mathbb{P}(X=9) = 0$$

while for $k \leq 8$ we there are $\binom{10-k}{2}$ subsets of $\{1,\ldots,10\}$ that have k as the smallest element. Hence

$$\mathbb{P}(X=k)=\frac{\binom{10-k}{2}}{120},\ \ 1\leq k\leq 8.$$

The mean of X is

$$\mathbb{E}[X]=\frac{1}{120}\sum_{k=1}^{8}k\binom{10-k}{2}=2.75.$$

Exercise 2.11 (a) In this case the number of tries $X\sim\mathcal{G}eom(1/5)$. Therefore $\mathbb{E}[X]=5$ and $\mathrm{var}(X)=20$.
(b) Now the number of tries X has range $\{1,\ldots,5\}$. For $k=1,2,\ldots,5$ we have

$$\mathbb{P}(X=k)=\frac{(4)_{k-1}}{(5)_k}=\frac{1}{5}.$$

Hence

$$\mathbb{E}[X]=\frac{1+\cdots+5}{5}=3,$$
$$\mathbb{E}[X^2]=\frac{1^2+\cdots+5^2}{5}=11,\ \ \mathrm{var}[X]=\mathbb{E}[X^2]-\mathbb{E}[X]^2=2.$$

Exercise 1

We call N_1, N_2 and N_3 the random variables for each question.

1. $N_1\sim\mathcal{B}er(2/1000)$. Of course $\mathbb{P}(N_1=5)=0$
2. $N_2\sim\mathcal{B}in(1000,0.001)$, so
$$\mathbb{P}(N_2=5)=\binom{1000}{5}\cdot 0.001^5\cdot 0.999^{995}\approx 0.305\%.$$
3. $N_3\sim\mathcal{N}eg\mathcal{B}in(1000,0.999)$, so $\mathbb{P}(N_3=5)=0$. For all $n\geq 1000$, we have
$$\mathbb{P}(N_3=n)=\binom{n}{1000}\cdot 0.001^{n-1000}\cdot 0.999^{1000}.$$

I might have not exactly thought the second part of the question through. Oh well.

Exercise 2

The number of strokes he needs to clear a given obstacle is the same as the first success in trying to put the ball in the hole. In other words, it is a random variable of distribution $\mathcal{G}eom(p)$. Since we know that the expectation is 1.5, we know that $1/p=1.5$, in other words $p=2/3$.

The number of strokes he needs to clear all 18 obstacles is the number of trials neeed to get 18 successes; in other words, it is a negative binomial of parameter $(18,2/3)$. The probability to finish in exactly 20 strokes is

$$\binom{20}{18}\cdot\left(\frac{2}{3}\right)^{18}\cdot\left(\frac{1}{3}\right)^{2}\approx 1.43\%.$$

Exercise 3

1. It is the first time there is no edge between two points: $\mathcal{G}eom(1-p)$.

   It doesn't matter if this doesn't count as a 'success' to us: if you want, you can imagine that you are playing against someone, and they are successful if you are stopped.

2. The event "we can go to infinity", call it E_∞, is included in the event "N is at least n", that we call E_n. As see in class, N is at least n if the first n experiments are 'failures', in our case if the first n edges are present, and this event has probability p^n. Hence, we have

   $$\mathbb{P}(E_\infty) \leq \mathbb{P}(E_n) = p^n.$$

   We can take the limit as n goes to infinity in the inequality, and deduce that $\mathbb{P}(E_\infty) \leq 0$, so this probability has to be zero.

3. The event E_∞ "we can go to infinity" is included in the union $E_+ \cup E_-$, where $E_\pm$ is the event "all edges from zero to $\pm\infty$ are present". According to the previous question, E_- and E_+ have probability zero.

   We can then either use Homework 1, and say that

   $$\mathbb{P}(E_\infty) \leq \mathbb{P}(E_+) + \mathbb{P}(E_-) = 0 + 0 = 0;$$

   alternatively, we can use the inclusion/exclusion formula:

   $$\mathbb{P}(E_\infty) = \mathbb{P}(E_+) + \mathbb{P}(E_-) - \mathbb{P}(E_+ \cap E_-) = -\mathbb{P}(E_+ \cap E_-) \leq 0.$$

   In any case, the probability cannot be positive, so it must be zero.