Homework 7
Solution

April 5th

Exercises from the book

Exercise 2.12 Let L denote the number of lids you need to lift until you find the hidden object. We have

$$\mathbb{P}(L=1)=\frac{1}{10},\ \ \mathbb{P}(L=2)=\frac{9}{10}\cdot\frac{1}{9}=\frac{1}{10},\ \ \mathbb{P}(L=3)=\frac{9}{10}\cdot\frac{8}{9}\cdot\frac{1}{8}=\frac{1}{10},$$

and similarly

$$\mathbb{P}(L=k)=\frac{1}{10},\ \ \forall k=1,2,\ldots,10,$$

so $L\sim\mathcal{U}nif(1,\ldots,10)$. Hence

$$\mathbb{E}[L]=\frac{1+10}{2}=5.5.$$

Exercise 2.14 If $X\sim\mathcal{B}in(n,p)$, then

$$\mathbb{E}[X]=np,\ \ \mathrm{Var}(X)=np(1-p)\Rightarrow 1-p=\frac{\mathrm{Var}(X)}{\mathbb{E}[X]}=\frac{0.9}{1}=0.9\Rightarrow p=0.1.$$

$$n=\frac{\mathbb{E}[X]}{p}=10.$$

$$\mathbb{P}(X>0)=1-\mathbb{P}(X=0)=1-(0.9)^{10}\approx 0.6513.$$

Exercise 2.16 We have $X\sim\mathcal{B}in(n=6,p=\frac{1}{6})$, $Y\sim\mathcal{B}in(n=12,p=\frac{1}{6})$. We have

$$\mathbb{E}[X]=1,\ \ \mathbb{E}[Y]=2$$

$$\mathbb{P}(X>1)=1-\mathbb{P}(X\leq 1)\approx 0.2632.$$

$$\mathbb{P}(Y>2)=1-\mathbb{P}(Y\leq 2)\approx 0.3225.$$

Exercise 1

1. By definition,

$$\begin{aligned}\mathbb{E}[X] &= -1 \cdot \mathbb{P}(X=-1) + 0 \cdot \mathbb{P}(X=0) + 0.4 \cdot \mathbb{P}(X=0.4) + 1 \cdot \mathbb{P}(X=1) \\ &= -\frac{1}{4} + 0 + \frac{2}{15} + \frac{1}{3} = \frac{13}{60} \approx 0.217,\end{aligned}$$

$$\mathbb{E}[X^2] = (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{12} + 0.4^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} = \frac{1}{4} + 0 + \frac{4}{75} + \frac{1}{3} = \frac{191}{300} \approx 0.637.$$

   In particular,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{191}{300} - \frac{169}{3600} = \frac{2123}{3600} \approx 0.590.$$

2. Similarly,

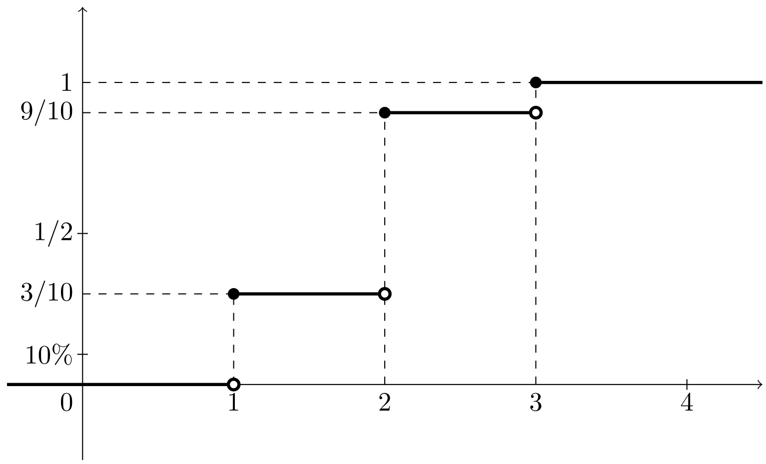
$$\begin{aligned}\mathbb{E}[\cos(5\pi X)] &= \cos(-5\pi) \cdot \frac{1}{4} + \cos(0) \cdot \frac{1}{12} + \cos(2\pi) \cdot \frac{1}{3} + \cos(5\pi) \cdot \frac{1}{3} \\ &= -\frac{1}{4} + \frac{1}{12} + \frac{1}{3} - \frac{1}{3} = \frac{1}{6} \approx 0.167.\end{aligned}$$

Exercise 2

The probability mass function of N is

$$\begin{aligned}p_N(0) &= \mathbb{P}(N=0) = 0, \\ p_N(1) &= \frac{\binom{3}{1}\binom{2}{2}}{\binom{5}{3}} = \frac{3}{10}, \\ p_N(2) &= \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}} = \frac{6}{10} = \frac{3}{5}, \\ p_N(3) &= \frac{\binom{3}{3}\binom{2}{0}}{\binom{5}{3}} = \frac{1}{10},\end{aligned}$$

and zero for any other value. This gives the following graph for the cumulative distribution function:



The median is 2 (the first x where the curve is at least 1/2), and the tenth percentile is 1 (the first x where the curve is at least 10%). If the level fell on a horizontal part of the graph (for instance the 90th percentile), then the quantile is the left endpoint of this part (for the 90th percentile case, 2).

Exercise 3

1. The expectation of N is
$$1 \cdot \mathbb{P}(N = 1) + \cdots + 6 \cdot \mathbb{P}(N = 6) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{7}{2}.$$

2. Using the usual formula for the variance,
$$\mathrm{Var}(N) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.92.$$

3. Let X be the sum of 100 independent variables of distribution $\mathcal{U}nif(\{1, \ldots, 6\})$. Then
$$\mathbb{E}[X] = 100 \cdot \mathbb{E}[N] = 350$$
by linearity of the expectation. Since the variables are independent, we can do the same for the variance:
$$\mathrm{Var}(X) = 100\mathrm{Var}(N).$$
The expected range of X is then more or less
$$350 \pm 10\sqrt{\frac{35}{12}} \approx 350 \pm 17.1.$$
The lower bound for this range is approximately 332.9. The sum of 319 then seems suspicious.

   It is difficult at this point of the class to tell how suspicious this number is. In the coming week and a bit, we will see that the probability to be this far from the expected value is at most 31%, which is still fairly high. During the last few classes of the course, we'll see that this probability is actually closer to 7%, which becomes more of a concern.

Exercise 4

1. We know that $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$. Moreover,

$$\begin{aligned}\mathbb{E}[XY] &= (-1)\cdot 1 \cdot \mathbb{P}(X=-1 \text{ and } Y=1) + (-1)\cdot(-1)\cdot \mathbb{P}(X=-1 \text{ and } Y=-1)\\ &\quad + 2\cdot y \cdot \mathbb{P}(X=2 \text{ and } Y=y)\\ &= -\frac{1}{3}+\frac{1}{3}+\frac{2y}{3} = \frac{2y}{3},\end{aligned}$$

$$\mathbb{E}[X] = -\frac{1}{3}-\frac{1}{3}+\frac{2}{3} = 0, \qquad \mathbb{E}[Y] = \frac{1}{3}-\frac{1}{3}+\frac{y}{3} = \frac{y}{3}.$$

   It means that $\mathrm{Cov}(X,Y) = 0$ if and only if $2y/3 = 0$. Hence we need $y = 0$.

2. If X and Y are independent, it means that Y has no influence on X. But

$$\mathbb{P}(X=-1) = \frac{2}{3},$$

   whereas

$$\mathbb{P}(X=-1|Y=-1) = \frac{\mathbb{P}(X=-1 \text{ and } Y=-1)}{\mathbb{P}(Y=-1)} = \frac{1/3}{1/3} = 1.$$

   So Y actually does influence X, and X and Y are not independent.

   If you prefer not to rely on conditional expectation, you can work with intersections of events:

$$\mathbb{P}(X=-1 \text{ and } Y=-1) = \frac{1}{3} \neq \frac{2}{9} = \mathbb{P}(X=-1)\cdot\mathbb{P}(Y=-1).$$