Homework 9
Solution

April 23rd

Exercise 2.31

(i) Since f is a probability density function, its integral must be one. In other words,

$$1 = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-1}^{1} \frac{c}{\sqrt{1-x^2}}\mathrm{d}x = c\left(\arcsin(1) - \arcsin(-1)\right) = c\pi.$$

From this we deduce

(ii) Since for $-1 \leq t \leq 1$, we have

$$\int_{-1}^{t} \frac{c}{\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\arcsin(t) + \frac{1}{2},$$

the distribution function is given by

$$F(t) = \begin{cases} \frac{1}{\pi}\arcsin(t) + \frac{1}{2} & -1 < t < 1, \\ 0 & t \leq -1, \\ 1 & t \geq 1. \end{cases}$$

Exercise 1

We have $(2x-1)^2 \geq 1/4$ if and only if $|2x-1| \geq 1/2$, which is the same as $|x - 1/2| \geq 1/4$. This happens for $x \leq 1/4$ or $x \geq 3/4$.

Using the definition of continuous random variables, we have

$$\begin{aligned} \mathbb{P}\big((2x-1)^2 \geq 1/4\big) &= \mathbb{P}(X \leq 1/4 \text{ or } X \geq 3/4) \\ &= \mathbb{P}(X \leq 1/4) + \mathbb{P}(X \geq 3/4) \\ &= \int_{-\infty}^{1/4} p_X(x)\mathrm{d}x + \int_{3/4}^{\infty} p_X(x)\mathrm{d}x. \end{aligned}$$

This would be true for any continuous random variable X. In our case, X is uniform over $[0,1]$, so we know that the density is one for $x \in [0,1]$, and zero everywhere else. So we can disregard the part where the density is zero, and we get

$$\mathbb{P}\big((2x-1)^2 \geq 1/4\big) = \int_{0}^{1/4} 1\mathrm{d}x + \int_{3/4}^{1} 1\mathrm{d}x = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Exercise 2

1. By definition,

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} p_X(u)\mathrm{d}u = C \int_{-\infty}^{x} \frac{\mathrm{d}u}{1+u^2} = C \cdot \left(\arctan(x) + \frac{\pi}{2}\right).$$

2. We know X but not Y, so we try to rephrase everything as properties of X:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X \leq y/2) = C \cdot \left(\arctan\left(\frac{y}{2}\right) + \frac{\pi}{2}\right).$$

3. Since

$$F_Y(y) = \int_{-\infty}^{y} p_Y(t)\mathrm{d}t,$$

we can use the second fundamental theorem of calculus to deduce

$$p_Y(y) = F'_Y(Y) = C \cdot \frac{\mathrm{d}}{\mathrm{d}y}\left(\arctan\left(\frac{y}{2}\right) + \frac{\pi}{2}\right) = \frac{C}{2 + y^2/2}.$$