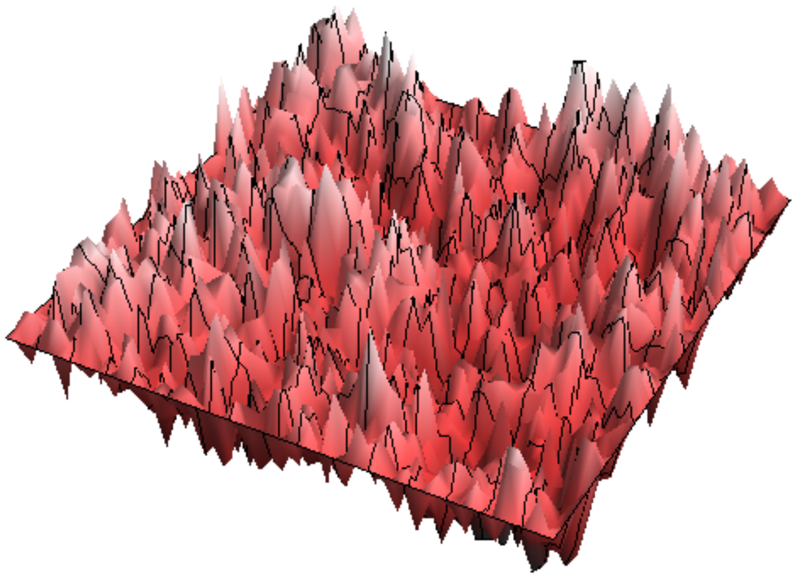
Constructive quantum field theory: an approach via the Brownian loop soup

Pierre Perruchaud
Université du Luxembourg

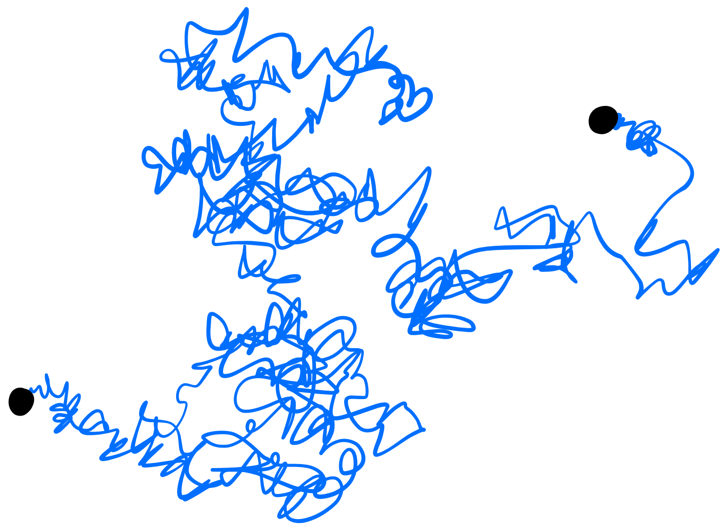
Joint work with Isao Sauzedde
University of Warwick

September 2024, *Interacting particles in the continuum,*
EURANDOM Eindhoven

A first example



A typical realization of the Gaussian free field
(from Wikipedia, Samuel S. Watson)



A Brownian bridge

A first example

Standard Gaussian free field: (almost) a random function ϕ defined on a domain of the plane.

Roughly, all $\phi(x)$ as well as the increments $\phi(x + dx) - \phi(x)$ are i.i.d Gaussian, conditioned to pasting globally to a function.

It is Gaussian, so it is determined by the covariance function $\mathbb{E}[\phi(x)\phi(y)]$.

A first example

A measure $\mathcal{E}_{x,y}$ on Brownian-like paths from x to y:

- run a Brownian motion $W : [0, \tau) \to M$ starting from x, killed with some rate and at the boundary;
- choose a time $U \in [0, \tau]$ with respect to Lebesgue;
- disintegrate according to W_U; specialize at y.

A first example

A measure $\mathcal{E}_{x,y}$ on Brownian-like paths from x to y:

- run a Brownian motion $W : [0, \tau) \to M$ starting from x, killed with some rate and at the boundary;
- choose a time $U \in [0, \tau]$ with respect to Lebesgue;
- disintegrate according to W_U; specialize at y.

Theorem.

For every $x, y \in M$, we have (weakly)

$$\mathbb{E}\left[\phi(x)\phi(y)\right] = \mathcal{E}_{x,y}[1].$$

Classical field theory

We work over spacetime.

Classical field theory

We work over spacetime.

Two fundamental objects:

- **a field ϕ,**
  i.e. a collection of $\phi(x)$ for every $x \in M$, where they all live in a different vector space E_x;

Classical field theory

We work over spacetime.

Two fundamental objects:

- **a field ϕ,**
  i.e. a collection of $\phi(x)$ for every $x \in M$, where they all live in a different vector space E_x;
- **a connection ∇,**
  which gives a way to compare $\phi(x)$ and $\phi(y)$ *given a path.*

Classical field theory

We work over spacetime.

Two fundamental objects:

- **a field ϕ,**
  i.e. a collection of $\phi(x)$ for every $x \in M$, where they all live in a different vector space E_x;
- **a connection ∇,**
  which gives a way to compare $\phi(x)$ and $\phi(y)$ *given a path.*

From $\gamma \in \text{path}(x, y)$ smooth, we get $\mathcal{H}ol^{\nabla}(\gamma) : E_x \to E_y$.
It sends concatenation to composition (functorial).

UNIVERSITÉ DU LUXEMBOURG

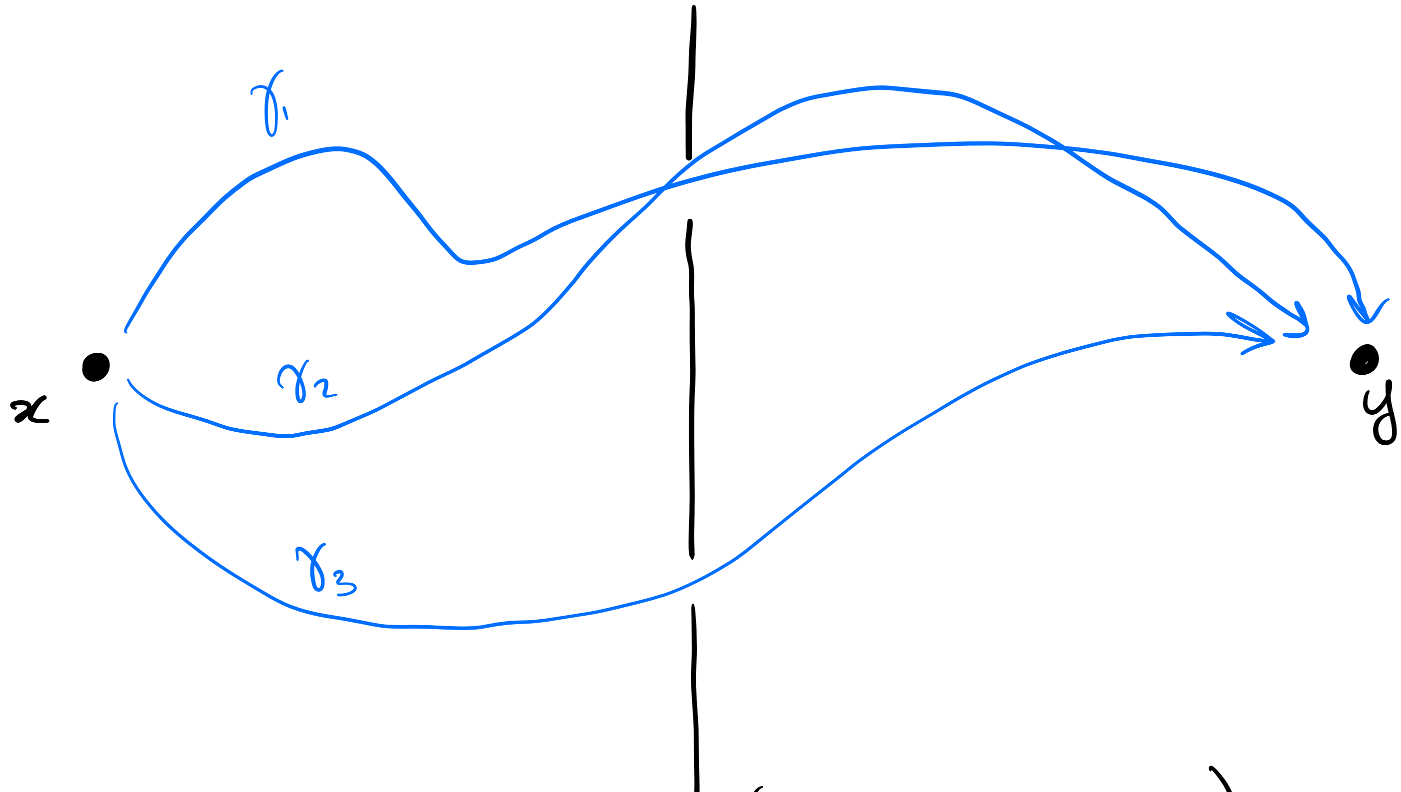
Classical field theory

- The field ϕ represents *matter* (particles!), for instance electrons.
- The connection ∇ represents *forces* (interactions!), for instance the electromagnetic potential.

$|\phi|$ large $\leftrightarrow$ large probability to detect particles

Classical field theory

- Field ϕ = matter (electrons)
- Connection ∇ = forces (electromagnetic potential)



$$\phi(y) = \underset{\substack{\gamma \\ x \to y}}{\operatorname{average}}\left(\mathrm{Hol}^{\nabla}(\gamma)\,\phi(x)\right)$$

A constructive quantum field theory

Our model: we want to construct a measure

$$\frac{1}{Z}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2-\lambda\|\phi\|_4^4+\mu\|\phi\|_2^2\Big)\mathcal{D}\phi\,\mathbb{P}^{YM}(\mathrm{d}\nabla).$$

- Some reference measure that we take for granted (Yang–Mills).

A constructive quantum field theory

Our model: we want to construct a measure

$$\frac{1}{Z}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2-\lambda\|\phi\|_4^4+\mu\|\phi\|_2^2\Big)\mathcal{D}\phi\mathbb{P}^{YM}(\mathrm{d}\nabla).$$

- Some reference measure that we take for granted (Yang–Mills).
- An interaction term. It gives a ∇-dependent H^1 norm.

A constructive quantum field theory

Our model: we want to construct a measure

$$\frac{1}{Z}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2-\lambda\|\phi\|_4^4+\mu\|\phi\|_2^2\Big)\mathcal{D}\phi\mathbb{P}^{YM}(\mathrm{d}\nabla).$$

- Some reference measure that we take for granted (Yang–Mills).
- An interaction term. It gives a ∇-dependent H^1 norm.
- A potential, in the famous sombrero shape.

A constructive quantum field theory

Our model: we want to construct a measure

$$\frac{1}{Z}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2-\lambda\|\phi\|_4^4+\mu\|\phi\|_2^2\Big)\mathcal{D}\phi\mathbb{P}^{YM}(\mathrm{d}\nabla).$$

- Some reference measure that we take for granted (Yang–Mills).
- An interaction term. It gives a ∇-dependent H^1 norm.
- A potential, in the famous sombrero shape.

Crucially, it is *not* a conditioning; under this distribution, ∇ does not follow $\mathbb{P}^{YM}$!

A constructive quantum field theory

Our model:

$$\frac{1}{Z}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2-(\lambda\|\phi\|_4^4-\mu\|\phi\|_2^2)\Big)\mathcal{D}\phi\,\mathbb{P}^{YM}(\mathrm{d}\nabla).$$

Reformulation:

$$\frac{Z_\nabla^{GFF}}{Z}\exp\Big(-(\lambda\|\phi\|_4^4-\mu\|\phi\|_2^2)\Big)\cdot\underbrace{\frac{1}{Z_\nabla^{GFF}}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2\Big)\mathcal{D}\phi}_{=:\mathbb{P}_\nabla^{GFF}(\mathrm{d}\phi)}\,\mathbb{P}^{YM}(\mathrm{d}\nabla)$$

A constructive quantum field theory

Our model:

$$\frac{1}{Z}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2-(\lambda\|\phi\|_4^4-\mu\|\phi\|_2^2)\Big)\mathcal{D}\phi\mathbb{P}^{YM}(\mathrm{d}\nabla).$$

Reformulation:

$$\frac{Z_\nabla^{GFF}}{Z}\exp\Big(-(\lambda\|\phi\|_4^4-\mu\|\phi\|_2^2)\Big)\cdot\underbrace{\frac{1}{Z_\nabla^{GFF}}\exp\Big(-\frac{1}{2}\|\nabla\phi\|_2^2\Big)\mathcal{D}\phi\,\mathbb{P}^{YM}(\mathrm{d}\nabla)}_{=:\mathbb{P}_\nabla^{GFF}(\mathrm{d}\phi)}$$

- We interpret $\mathbb{P}^{GFF}$ as the Gaussian free field.
- We must have $Z=\mathbb{E}^{YM}[Z_\nabla^{GFF}]$.

A constructive quantum field theory

Context:

- A Riemannian manifold (M, g), compact with boundary; a mass function $m : M \to \mathbb{R}_+$.
- A complex vector bundle E, with a Hilbert metric.
- A section $\phi : M \to E$ and a metric connection ∇ on E (TBA).

Main goal: to understand *with loops* (i.e. compute expectations under) the variables

$$\int |\phi(x)|^4 \mathrm{d}x, \qquad \int |\phi(x)|^2 \mathrm{d}x \qquad \text{and} \qquad \frac{Z^{GFF}_{\nabla}}{\widetilde{\mathbb{E}}^{YM}[Z^{GFF}_{\widetilde{\nabla}}]}$$

under the distribution $\mathbb{P}^{GFF}_{\nabla}(\mathrm{d}\phi)\mathbb{P}^{YM}(\mathrm{d}\nabla)$.

Twisted Gaussian free field

The Gaussian free field ϕ twisted by ∇ of mass m is the Gaussian field with Cameron-Martin bracket

$$Q(\zeta, \xi) = \int_M \left(\langle \nabla \zeta, \nabla \xi \rangle + \langle \zeta, m\xi \rangle \right).$$

- There is some tension between the values, pushing ϕ to be continuous.
- In dimension 1, it is a Brownian bridge ($m = 0$, $\nabla = \partial_x$).
- In dimension 2, it just fails to be a function: it is a measure. In general, it is almost $H^{1-d/2}$.
- It is conformally invariant in dimension 2, hence fractal.
- It is well understood.

Twisted Gaussian free field

First apparition of the loops: the $2k$-point functions.

We want to compute the correlation of the values of ϕ at different points. At x, we can look at the coordinate of $\phi(x)$ in the direction of $v \in E_x$. Let us write it $\langle v, \phi \rangle$.

Twisted Gaussian free field

First apparition of the loops: the $2k$-point functions.

We want to compute the correlation of the values of ϕ at different points. At x, we can look at the coordinate of $\phi(x)$ in the direction of $v \in E_x$. Let us write it $\langle v, \phi \rangle$.

Theorem.

For all $v_i \in E_{x_i}$, we have (weakly)

$$\mathbb{E}\left[\langle v_1, \phi\rangle \overline{\langle v_2, \phi\rangle} \cdots \langle v_{2k-1}, \phi\rangle \overline{\langle v_{2k}, \phi\rangle}\right] = \sum_{\sigma \in \mathfrak{S}_k} \prod_{i=1}^{k} \mathcal{E}_{x_{2i+1}, x_{2\sigma(i)}}\left[\langle v_j, \mathcal{H}ol^{\nabla}(\ell) v_i\rangle\right].$$

$\mathcal{E}_{x,y}$ is a measure over Brownian-like paths from x to y (see introduction).

Determinants of Laplacians

We discuss Z_∇^{GFF}. For a Gaussian measure of the form

$$\frac{1}{Z}\exp\left(-\frac{1}{2}(u^*Qu)\right)du,$$

we have

$$Z = \sqrt{(2\pi)^{\text{dimension}} / \det Q}.$$

Determinants of Laplacians

We discuss Z_{∇}^{GFF}. For a Gaussian measure of the form

$$\frac{1}{Z} \exp\Big(-\frac{1}{2}(u^* Q u)\Big) \mathrm{d}u,$$

we have

$$Z = \sqrt{(2\pi)^{\text{dimension}} / \det Q}.$$

In our case, we should consider

$$Z_{\nabla}^{GFF} = \sqrt{(2\pi)^{\infty} / \det(\tfrac{1}{2}\nabla^* \nabla + m)}.$$

Determinants of Laplacians

We discuss Z_{∇}^{GFF}. For a Gaussian measure of the form

$$\frac{1}{Z} \exp\left(-\frac{1}{2}(u^* Q u)\right) \mathrm{d}u,$$

we have

$$Z = \sqrt{(2\pi)^{\text{dimension}} / \det Q}.$$

In our case, we should consider

$$Z_{\nabla}^{GFF} = \sqrt{(2\pi)^{\infty} / \det(\tfrac{1}{2}\nabla^*\nabla + m)}.$$

We are interested in the dependence on ∇:

$$\frac{Z_{\nabla_1}^{GFF}}{Z_{\nabla_0}^{GFF}} = \left(\frac{\det(\frac{1}{2}\nabla_0^*\nabla_0 + m)}{\det(\frac{1}{2}\nabla_1^*\nabla_1 + m)} \right)^{1/2}.$$

Determinants of Laplacians

The kth eigenvalue of the Laplacian is $k^{2/d+o(1)}$, $d = \dim M$.
The product of the eigenvalues is very ill-defined.

A theory of determinants of operators: ζ-regularization.

$$\zeta_\nabla(z) := \sum_{\lambda \in \text{spectrum}\left(\frac{1}{2}\nabla^*\nabla + m\right)} \lambda^{-z}$$

converges for $\Re z > d/2$, and extends meromorphically to $\mathbb{C}$.
We set

$$\det(\tfrac{1}{2}\nabla^*\nabla + m) := \exp\left(-\zeta'_\nabla(0)\right).$$

Determinants of Laplacians

There exists a measure Λ on loops in M such that the following holds.

Theorem (P. Sauzedde).

Suppose

- M has dimension 2 or 3;
- either the mass does not identically vanish, or we have a boundary.

Then the determinant rewrites as

$$\det\left(\tfrac{1}{2}\nabla^*\nabla + m\right)^{-1} = \det(\Delta_M)^{-\operatorname{rk} E} \cdot \mathbb{E}^{BLS}\left[\prod_{\ell\in\mathcal{L}} \operatorname{tr}\mathcal{H}ol^{\nabla}(\ell)\right],$$

where $\mathcal{L}$ is a Poisson process of loops of intensity Λ (Brownian loop soup).

Determinants of Laplacians

The Brownian loop soup: Poisson process of loops with intensity

$$\Lambda(\ell \in \mathrm{d}\ell) = \int_M \mathcal{E}_{x,x}\left[|\ell| \mathbf{1}_{\ell \in \mathrm{d}\ell}\right] \mathrm{d}x,$$

where $\mathcal{E}_{x,y}$ is the measure on paths from x to y of the introduction, and $|\ell|$ is the duration of the path ℓ.

Determinants of Laplacians

The Brownian loop soup: Poisson process of loops with intensity

$$\Lambda(\ell \in \mathrm{d}\ell) = \int_M \mathcal{E}_{x,x}\left[|\ell| \mathbf{1}_{\ell \in \mathrm{d}\ell}\right] \mathrm{d}x,$$

where $\mathcal{E}_{x,y}$ is the measure on paths from x to y of the introduction, and $|\ell|$ is the duration of the path ℓ.

- It is conformally invariant, hence very fractal in nature.
- We always have infinitely many small loops.
- We have infinitely large loops, unless there is some killing process (boundary, mass).

The key observation

Key observation: they are both directly linked to the heat kernels K and $\hat{p}$:

$$\mathbb{E}^{GFF}[\phi(v)\phi(w)] = \int_0^\infty \langle v, K_t(x,y)w\rangle \mathrm{d}t,$$

$$\mathcal{E}_{x,y}(\ell \in \mathrm{d}\ell) = \int_0^\infty \hat{p}_t(x,y)\mathbb{E}_{t,x,y}\left[\ell \in \mathrm{d}\ell\right]\mathrm{d}t$$

for K the kernel of $\frac{1}{2}\nabla^*\nabla + m$, $\hat{p}$ the base, massless, boundaryless heat kernel and $\mathbb{E}$ the massive, boundary-killed Brownian bridge.

The key observation

Key observation: they are both directly linked to the heat kernels K and $\hat{p}$:

$$\mathbb{E}^{GFF}[\phi(v)\phi(w)] = \int_0^\infty \langle v, K_t(x,y)w\rangle \mathrm{d}t,$$

$$\mathcal{E}_{x,y}(\ell \in \mathrm{d}\ell) = \int_0^\infty \hat{p}_t(x,y)\mathbb{E}_{t,x,y}\left[\ell \in \mathrm{d}\ell\right]\mathrm{d}t$$

for K the kernel of $\frac{1}{2}\nabla^*\nabla + m$, $\hat{p}$ the base, massless, boundaryless heat kernel and $\mathbb{E}$ the massive, boundary-killed Brownian bridge.

It turns out that

$$K_t(x,y) = \hat{p}_t(x,y)\mathbb{E}_{t,x,y}\left[\mathrm{tr}\,\mathcal{H}ol^{\nabla}(\ell)^{-1}\right]$$

Some more work

- The term ϕ^2 is related to the integral of the mass along loops.
- The term ϕ^4 should be related to the total self-intersection of the loop soup.
- The interaction between the terms is less obvious than it looks...
- Can we say anything in dimension 4?

Some more work

- The term ϕ^2 is related to the integral of the mass along loops.
- The term ϕ^4 should be related to the total self-intersection of the loop soup.
- The interaction between the terms is less obvious than it looks...
- Can we say anything in dimension 4?

Thank you for your attention

From the heat equation to the determinant

$$
\begin{aligned}
\zeta_\nabla(z) &= \sum_\lambda \lambda^{-z} \\
&= \sum_\lambda \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{e}^{-\lambda t} \frac{\mathrm{d}t}{t^{1-z}} \\
&= \frac{1}{\Gamma(z)} \int_0^\infty \operatorname{Tr} \mathrm{e}^{-tL} \frac{\mathrm{d}t}{t^{1-z}} \\
&= \frac{r}{\Gamma(z)} \int_0^\infty \int_M \operatorname{tr} K_t(x,x)\, \mathrm{d}x \frac{\mathrm{d}t}{t^{1-z}} \\
&= \frac{r}{\Gamma(z)} \int_0^\infty \int_M \hat{p}_t(x,x) \mathbb{E}_{t,x,x}[\operatorname{tr} \mathcal{H}ol^\nabla(W)]\, \mathrm{d}x \frac{\mathrm{d}t}{t^{1-z}} \\
&= \frac{r}{\Gamma(z)} \int |\ell|^z \operatorname{tr} \mathcal{H}ol^\nabla(\ell) \Lambda(\mathrm{d}\ell)
\end{aligned}
$$

Multiplicative Campbell

$$
\begin{aligned}
\mathbb{E}^{BLS}\Big[\prod_{\ell\in\mathcal{L}}(1+h)(\ell)\Big] &= e^{-|\Lambda|}\sum_{k\geq 0}\frac{|\Lambda|^k}{k!} \\
&\quad \cdot \left(\frac{\Lambda}{|\Lambda|}\right)^{\otimes k}\big((1+h)(\ell_1)\cdots(1+h)(\ell_k)\big) \\
&= e^{-|\Lambda|}\exp(\Lambda(1+h)) \\
&= \exp\big(\Lambda(h)\big)
\end{aligned}
$$