Differential topology for dynamical random fields

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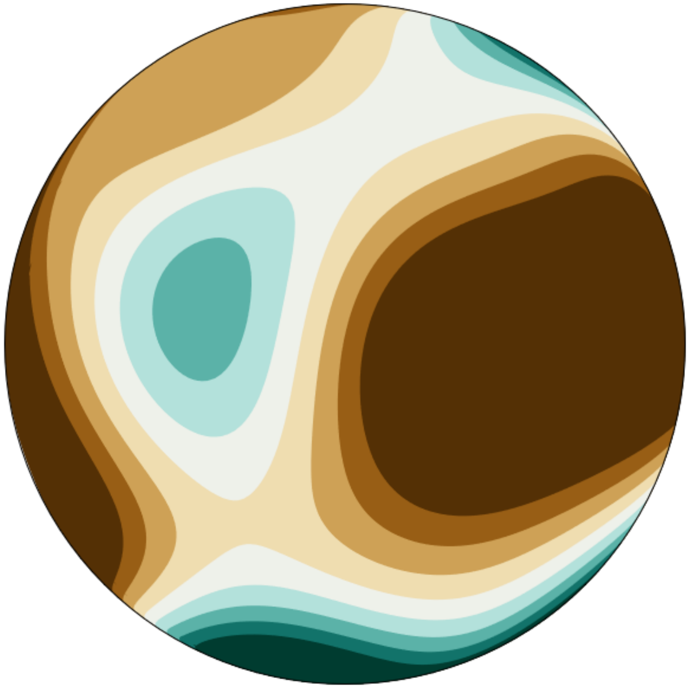
Brownian nodal submanifolds

Main object: random smooth functions f_t that vary with respect to some time parameter t.
Main question: how does the zero set $\mathcal{Z}_t$ evolve?



Example: $x \mapsto f_t(x) \in \mathbb{R}$ smooth on the sphere for t fixed, all the $t \mapsto f_t(x)$ jointly Brownian.

Brownian nodal submanifolds

For $t > 0$ fixed, what can we say about the zero set $\mathcal{Z}_t$?

Half-theorem.

In most situations $\mathcal{Z}_t$ is a collection of disjoint smooth loops.

In other words,

$$\forall t > 0, \mathbb{P}(\mathcal{Z}_t \text{ is a submanifold}) = 1.$$

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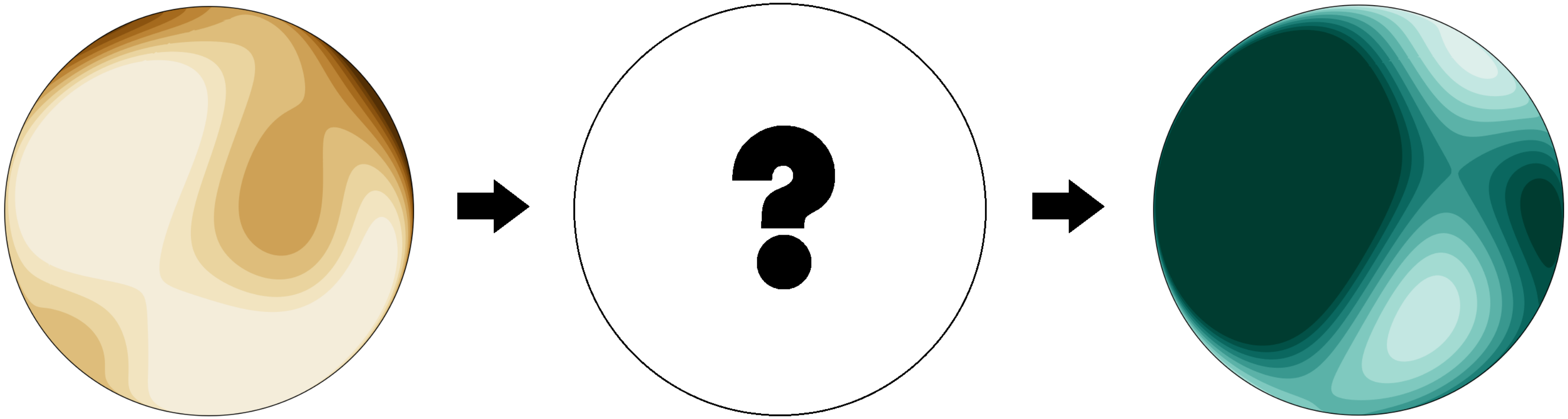
When $t > 0$ varies, is it the same?
By Fubini,

$$\mathbb{P}(\{t > 0 : \mathcal{Z}_t \text{ is a submanifold}\} \text{ has zero measure}) = 1.$$

But do we actually have exceptional times?

Brownian nodal submanifolds

Under reasonable hypotheses, we *must* have exceptional times where the topology changes.



From $f_s > 0$ *to* $f_t < 0$*, we must create a point in* $\mathcal{Z}_t$ *somewhere*

At (some of) those times, $\mathcal{Z}_t$ will not be a submanifold.

Brownian nodal submanifolds

We say that f_t is *nice* if it does not have a critical zero: there is no point x with $f(x) = 0$ and $\mathrm{d}f_x = 0$.

If f_t is nice, then $\mathcal{Z}_t$ is a collection of disjoint smooth loops.

Brownian nodal submanifolds

We say that f_t is *nice* if it does not have a critical zero: there is no point x with $f(x) = 0$ and $\mathrm{d}f_x = 0$.

If f_t is nice, then $\mathcal{Z}_t$ is a collection of disjoint smooth loops. Moreover, f_s is nice for $s \approx t$ and we can deform $\mathcal{Z}_t$ into $\mathcal{Z}_s$.

By the above reasoning, there must exist exceptional times $t > 0$ where f_t is *not* nice.

Question.

What can we say about $\mathcal{Z}_s$ for $s \approx t$ when f_t is not nice?
Can we say anything about the set of exceptional times?

Brownian nodal submanifolds

Let M be a closed manifold, $E \to M$ a vector bundle, F a Banach space of smooth sections, $t \mapsto f_t$ a Brownian motion of full support with values in F.

Example: Brownian functions $\mathbb{S}^2 \to \mathbb{R}$ written as

$$t, (x, y, z) \mapsto \sum_{n \geq 0} c_n \sum_{k_x + k_y + k_z = n} W_t^{(k)} x^{k_x} y^{k_y} z^{k_z}$$

for $W^{(k)}$ independent standard Brownian motions and $(c_n)_{n \geq 0}$ decreasing fast enough.

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Under reasonable hypotheses,

I. we can describe $\mathcal{Z}_s$ around the exceptional times t,

II. we can describe the set

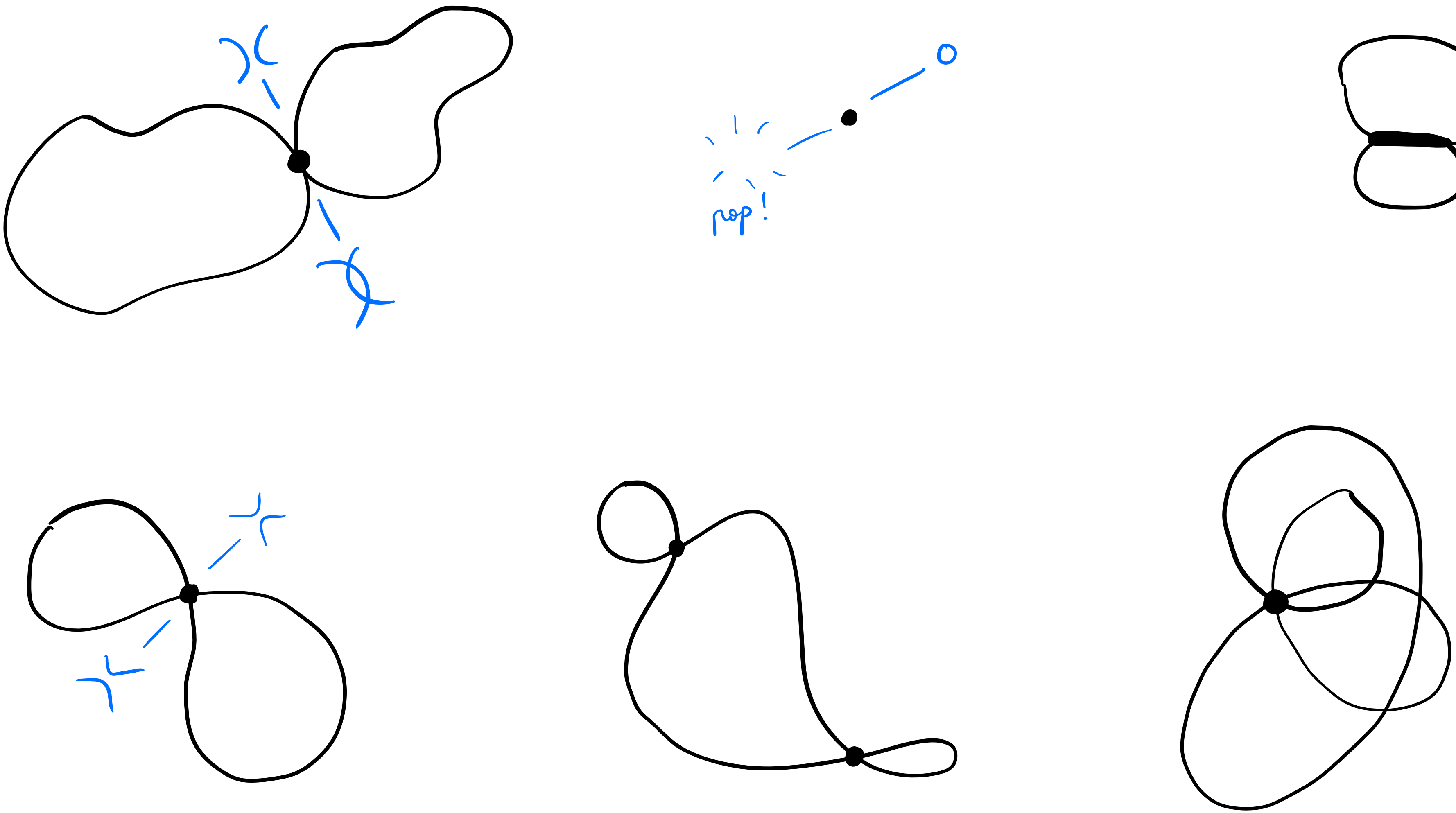
$$\{t > 0 : t \text{ is exceptional}\} \subset (0, \infty).$$

I. Structure of the discriminant set

Which singularities *can* we get? Which singularities *do* we get?

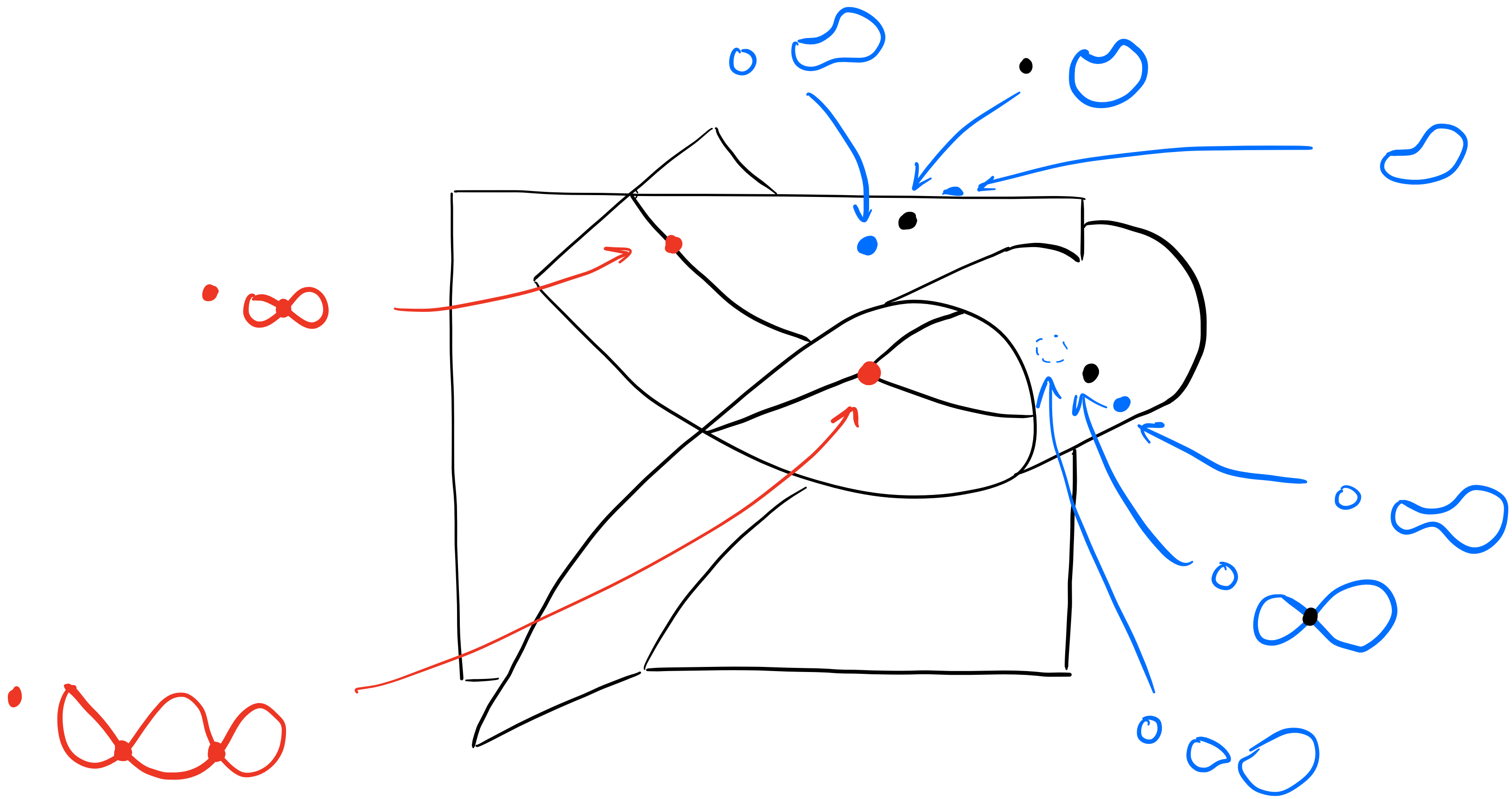
Structure of the discriminant set

Which type of singularities can we have?



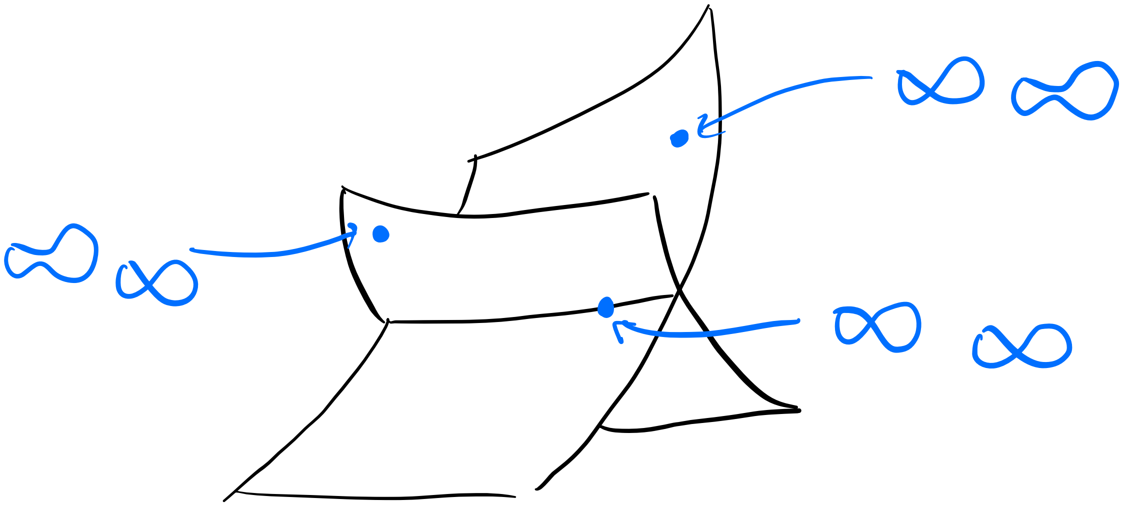
Structure of the discriminant set

We denote by $\Delta \subset F$ the set of non-nice sections (zero is not regular). It is the central object of our study, we call it the *discriminant set.* It looks a bit like this:



The discriminant set inside the infinite-dimensional space of functions F

Structure of the discriminant set



The singularities on the hypersurface look a bit like cones, and this is indeed what they are.

Definition.

A Morse function on M is a function with a single critical zero, given locally by

$$(x_1, \ldots, x_d) \mapsto \left(x_1, \ldots, x_r, \pm|x_{r+1}|^2 \pm |x_{r+2}|^2 \pm \cdots \pm |x_d|^2\right).$$

Structure of the discriminant set

Theorem (P.-Stecconi).

Suppose that the vectors

$$\big(f_1(x), \mathrm{d}(f_1)_x, \mathrm{Hess}(f_1)_x\big) \quad \text{and} \quad \big(f_1(x), \mathrm{d}(f_1)_x, f_1(y), \mathrm{d}(f_1)_y\big) \qquad \textbf{(H)}$$

are non-degenerate for all $x, y \in M$. Then

$$\Delta = \Delta_{\text{Morse}} \cup \Delta_{\text{residual}},$$

where

- Δ_{Morse} is the surface of all Morse functions;
- Δ_{residual} is a complicated object of codimension 2.

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where

- Δ_{Morse} is the surface of all Morse functions;
- Δ_{residual} is a complicated object of codimension 2.

Remark. Δ_{residual} cannot be completely peeled in strata of decreasing regularity; for instance, the order of tangency of two curves can increase to infinity, but also *be* infinite.

II. Infinite-dimensional Brownian motion

How does $t \mapsto f_t$ interact with Δ?

Infinite-dimensional Brownian motion

What seems reasonable for $t \mapsto \mathcal{Z}_t$ from the picture:

- Δ_{residual} is never touched by Brownian motion, i.e. the singularities are at most Morse;
- Brownian motion *does* touch Δ_{Morse} sometimes, and it oscillates between the two sides of the hypersurface.
  This corresponds to $\mathcal{Z}_t$ oscillating between the two resolutions of the singularity.

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Theorem (P.-Stecconi).

This is all true.

Infinite-dimensional Brownian motion

We avoid bad singularities

Theorem.

In finite dimension, Brownian motion avoids objects of codimension $c > 2$.

Proof.

If W_t is in X, then $W_{k/N}$ is about $O(N^{-1/2})$ away from it.

$$\begin{aligned}
\mathbb{P}(W_{|[0,1]} \text{ touches } X) &\leq \mathbb{P}\left(\exists k, d(W_{k/N}, X) < N^{-1/2+\varepsilon}\right) + o(1) \\
&\leq N \cdot \sup_k \mathbb{P}\left(W_{k/N} \in X + B_0(N^{-1/2+\varepsilon})\right) \\
&\approx N \cdot (N^{-1/2+\varepsilon})^{\operatorname{codim} X},
\end{aligned}$$

This goes to zero when $\operatorname{codim} X > 2$. □

Infinite-dimensional Brownian motion

We avoid bad singularities

Theorem (P.-Stecconi).

In F, Brownian motion avoids "*trails*" of codimension $c > 2$ and *submanifolds* of codimension $c \geq 2$.

Proof.

Trails are regular enough to make the finite-dimensional proof work, and irregular enough for our purposes.
For submanifolds, the proof is subtle, but follows from finite-dimensional results. □

Theorem (P.-Stecconi).

Under $(\mathbf{H})$, Δ_{residual} decomposes as a submanifold of codimension 2 and a trail of codimension 3.

Infinite-dimensional Brownian motion

We oscillate around Morse singularities

Theorem (P.-Stecconi).

Suppose we hit Δ_{Morse} at time t.

Locally, there is a nice, somewhat explicit semimartingale $s \mapsto A_s$ such that

- $A_0 = 0$, and $A_s = 0$ if and only if $\mathcal{Z}_{t+s}$ has a Morse singularity;
- $A_s > 0$ if and only if $\mathcal{Z}_{t+s}$ is resolved in one way;
- $A_s < 0$ if and only if $\mathcal{Z}_{t+s}$ is resolved in a second way.

Infinite-dimensional Brownian motion

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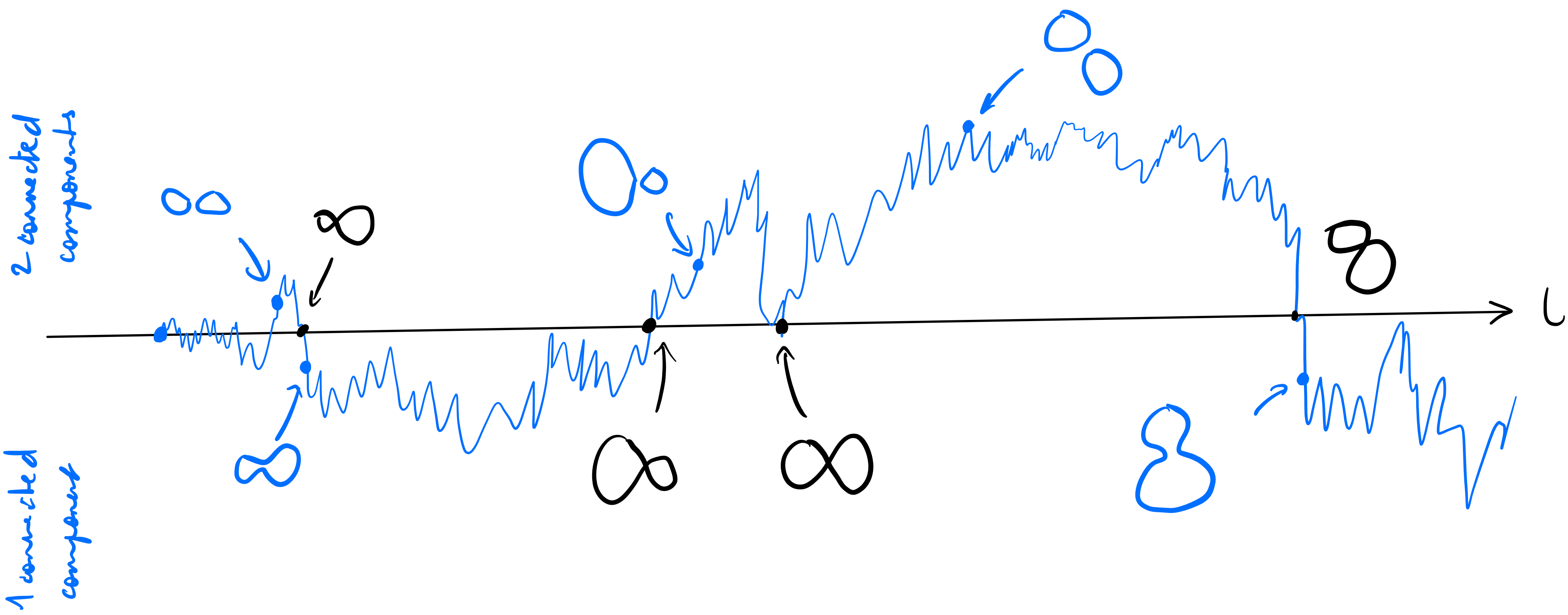
Theorem (P.-Stecconi).

Locally, there is a semimartingale $s \mapsto A_s$ that drives the singularities of $\mathcal{Z}_t$.



Infinite-dimensional Brownian motion

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Theorem (P.-Stecconi).

Locally, there is a semimartingale $s \mapsto A_s$ that drives the singularities of $\mathcal{Z}_t$.

Corollary.

The set of exceptional times in $[0, 1]$ is either empty, or a Cantor set of Hausdorff dimension $1/2$.

Infinite-dimensional Brownian motion

A blueprint for the study of $\mathcal{Z}_t$

Philosophy: if we want to understand $\mathcal{Z}_t$ through some invariants (volume, number of connected components, total curvature, Euler characteristic, diameter...), we only need to understand how it behaves under continuous deformation and Morse surgery.

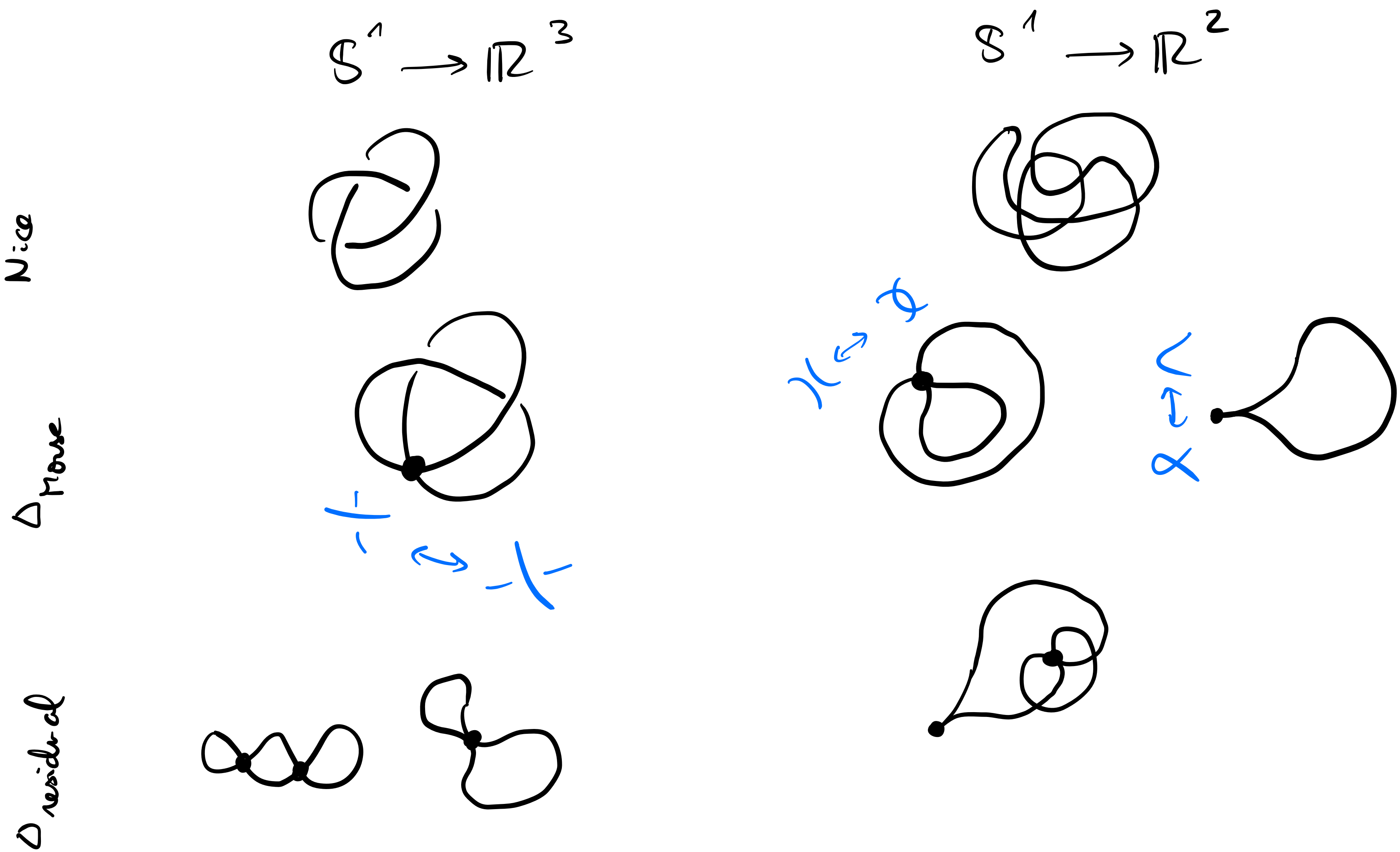
Theorem (P.-Stecconi).

Under (**H**), the volume of the nodal set is $(1/4 - \varepsilon)$-Hölder if it has dimension at least one.

III. More topological features, more randomness!

Extensions

Other topological objects



Extensions

Other types of randomness

Under (**H**), all these processes avoid objects of codimension 2:

- Coordinate-wise Fractional Brownian motion with $H > 1/2$
- Coordinate-wise Rosenblatt process*
- Solution to the heat equation with random enough initial condition

Under (**H**), all these processes avoid objects of codimension 3 and submanifolds of codimension 2:

- Solutions to $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t$ for b of finite rank
- Coordinate-wise stochastic integrals $t \mapsto \int_0^t h_s \mathrm{d}W_s$*
- Ornstein–Uhlenbeck processes*

* 98% confidence but no full proof

Conclusion

How to prove that for a process $t \mapsto f_t$, some geometric object $t \mapsto \mathcal{Z}_t$ is topologically nice except for sparse times where it is topologically not too bad:

- Define the space $\Delta = \Delta_{\text{Morse}} \cup \Delta_{\text{residual}}$ of f that are not nice, decomposed into “not too bad” and “actually bad”
- Show that Δ_{Morse} is a hypersurface
- Show that Δ_{residual} decomposes into a submanifold of codimension 2, and an object of codimension 3
- Prove a structure result on the set of times where $t \mapsto f_t$ hits hypersurfaces
- Prove that $t \mapsto f_t$ does not hit submanifolds of codimension 2 nor objects of codimension 3

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Thank you for your attention.